

—Chapter 5—

Special Relativity and The Fields of Constant Moving Charges

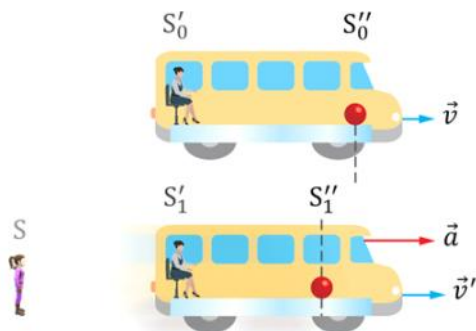
5-1 Einstein's Postulations in Special Relativity

A. INERTIAL FRAME OF REFERENCE

- (1) Inertia refers to the tendency of an object to remain at rest or continue moving in a straight line at a constant velocity. Newton provided a quantitative explanation of inertia and formulated it as the first law of motion:

A body at rest remains at rest or, if in motion, remains in motion at constant velocity unless acted on by a net external force.

- (2) Consider a scenario that an observer is outside the bus, and an observer and a red ball are inside the bus.

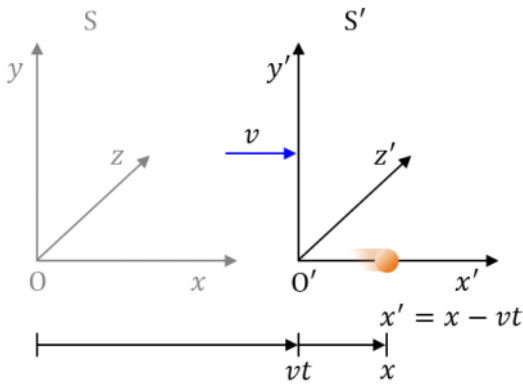


Frames S , S'_0 , and S''_0 are inertial frames of reference where Newton's first law applies (rest or uniform motion if without external force), i.e., if we have frame of references (S'_0 and S''_0) that are moving relative to an inertial frame (S) with uniform relative velocities ($\vec{v}$), then these frames are called inertial frame of references.

Suppose you know that no forces act on S'_1 and S''_1 . If S'_1 and S''_1 appear to be accelerating from your perspective, reference frames that represents your perspective are not inertial frames.

B. GALILEAN TRANSFORMATION IN EUCLIDEAN SPACE

- (1) Consider two inertial frames in Euclidean space. The frame S' is moving forward with constant velocity v relative to the frame S .



The relationship between positions in two inertial frames:

$$\left. \begin{array}{l} x' = x - vt \\ y' = y \\ z' = z \\ t' = t \end{array} \right\} \dots \text{Galilean transformation}$$

- (2) In classical mechanics era, Galilean transformation is a fundamental law. Thus we could deduce the relative velocity (addition of velocities) and the invariant of Newton's equation.

1. Relative velocity

$$\frac{dx'}{dt'} = \frac{d}{dt'}(x - vt) = \frac{dx}{dt'} - \frac{vdt}{dt'}$$

Since $dt' = dt$, we obtain

$$\frac{dx'}{dt'} = \frac{dx}{dt} - \frac{vdt}{dt} = \frac{dx}{dt} - v \Rightarrow u' = u - v$$

2. Invariant of Newton's equation

$$F' = \frac{dp'}{dt'} = m \frac{du'}{dt'} = m \frac{d}{dt'}(u - v) = m \frac{du}{dt'}$$

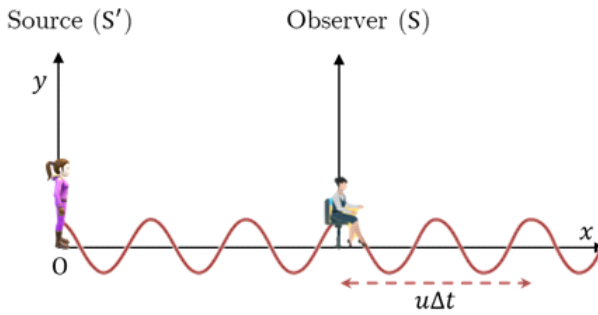
Since $dt' = dt$, we obtain

$$F' = m \frac{du'}{dt'} = m \frac{du}{dt} = F$$

EXAMPLES:

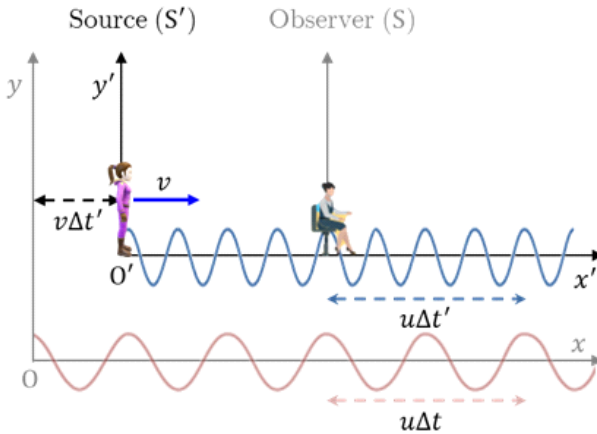
1. Doppler effect provides a direct evidence to exhibit the validation of Galilean transformation.

ANSWER:



$$N\lambda = u\Delta t \Rightarrow f = \frac{u}{\lambda} = \frac{N}{\Delta t}$$

Considering the source approaches observer



$$N\lambda' + v\Delta t' = u\Delta t' \Rightarrow \lambda' = \frac{u\Delta t' - v\Delta t'}{N} = \frac{u - v}{N} \Delta t'$$

$$f' = \frac{u}{\lambda'} = \frac{u}{\frac{(u - v)}{N} \Delta t'} = \frac{u}{u - v} \frac{N}{\Delta t'} = \frac{u}{u - v} \frac{f \Delta t'}{\Delta t'}$$

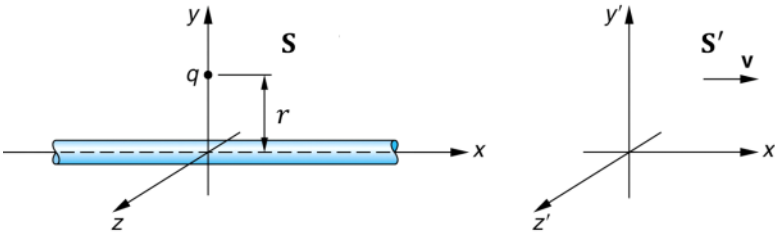
Since $\Delta t = \Delta t'$, we obtain

$$f' = \frac{u}{u - v} f$$

C. EINSTEIN'S POSTULATIONS

(1) Lorentz forces of constant moving charges

Consider an infinity long wire carrying uniform charge λ and a test charge $+q$ at r .



In S frame:

$$\oint \vec{E} \cdot d\vec{a} = E \cdot 2\pi r l = \frac{Q}{\epsilon_0} \Rightarrow \vec{E} = \frac{Q}{2\pi\epsilon_0 r l} = \frac{\lambda}{2\pi\epsilon_0 r} \hat{r}$$

$$\vec{B} = 0$$

$$\vec{F}_y = q\vec{E} + q\vec{v} \times \vec{B} = \frac{q\lambda}{2\pi\epsilon_0 r} \hat{y}$$

In S' frame:

$$\vec{E}' = \vec{E} = \frac{\lambda'}{2\pi\epsilon_0 r'} \hat{r}'$$

$$\oint \vec{B}' \cdot d\vec{l}' = B' \cdot 2\pi r' = \mu_0 I = \mu_0 \lambda' v \Rightarrow \vec{B}' = -\frac{\mu_0 \lambda' v}{2\pi r'} \hat{z}'$$

$$\vec{F}'_y = q\vec{E}'_y + qvB'\hat{y}' = \frac{q\lambda'}{2\pi\epsilon_0 r} \hat{y}' - \frac{\mu_0 q\lambda' v^2}{2\pi r} \hat{y}'$$

Since

$$c^2 = \frac{1}{\mu_0 \epsilon_0}$$

we obtain

$$\vec{F}'_y = \frac{q\lambda'}{2\pi\epsilon_0 r} \hat{y}' - \frac{q\lambda' v^2}{2\pi\epsilon_0 r c^2} \hat{y}' = \frac{q\lambda'}{2\pi\epsilon_0 r} \left(1 - \frac{v^2}{c^2}\right) \hat{y}'$$

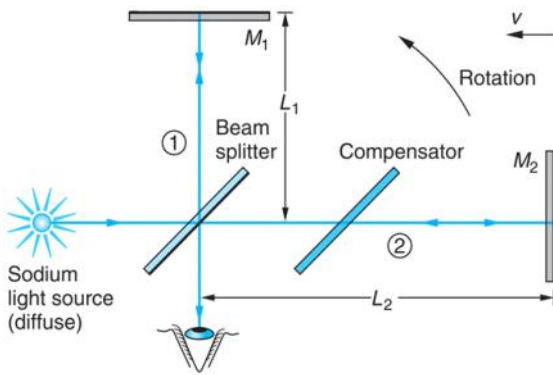
$$\Rightarrow \vec{F}_y \neq \vec{F}'_y$$

Thus, the Lorentz force are different in one inertial frame at rest and one inertial frame with constant moving velocity.

(2) Michelson-Morley Experiment

OS:

Nineteenth-century physicists thought reasonably at the time that light must propagate in the ether analogously to the way that sound propagates in a material medium such as air. Maxwell asked whether the velocity of Earth relative to the ether might affect the observed speed of light.



v along L_2

$$t_2 = \frac{L_2}{c-v} + \frac{L_2}{c+v} = \frac{2L_2}{c} \frac{1}{1-v^2/c^2}$$

$$t_1 = \frac{2L_1}{\sqrt{c^2 - v^2}} = \frac{2L_1}{c} \frac{1}{\sqrt{1-v^2/c^2}}$$

$$\Delta t = t_2 - t_1 = \frac{2L_2}{c} \frac{1}{1-v^2/c^2} - \frac{2L_1}{c} \frac{1}{\sqrt{1-v^2/c^2}}$$

v along L_1 (rotate 90°)

$$\Delta t' = t'_2 - t'_1 = \frac{2L_2}{c} \frac{1}{\sqrt{1-v^2/c^2}} - \frac{2L_1}{c} \frac{1}{1-v^2/c^2}$$

Path difference $\delta x = c\delta t$

$$\delta t = \Delta t - \Delta t' = \frac{2(L_1 + L_2)}{c} \left[\frac{1}{1-v^2/c^2} - \frac{1}{\sqrt{1-v^2/c^2}} \right]$$

As $v \ll c$

$$\delta t \approx \frac{2(L_1 + L_2)}{c} \left[\left(1 + \frac{v^2}{c^2} + \dots \right) - \left(1 + \frac{v^2}{2c^2} + \dots \right) \right] = \frac{v^2}{c^3} (L_1 + L_2)$$

Phase difference $\Delta\phi$

$$\Delta\phi = 2\pi \frac{\delta x}{\lambda} = 2\pi \frac{c\delta t}{\lambda} = \frac{2\pi v^2}{\lambda c^2} (L_1 + L_2)$$

Observe the shift in the interference pattern.

Experimental results:

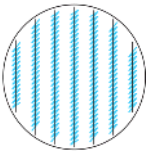
$$v = 3 \times 10^4 \text{ m/s}$$

$$L_1 = L_2 = 1.2 \text{ m}$$

$$\lambda = 600 \text{ nm}$$

$$\Rightarrow \Delta\phi = \frac{2\pi(3 \times 10^4)^2}{600 \times 10^{-9} \times (3 \times 10^8)^2} (1.2 + 1.2) \approx 0.08\pi$$

Width of fringe $d = \delta x / \lambda = 0.04$ is detectable



The interference pattern was not changed after rotation, i.e.,

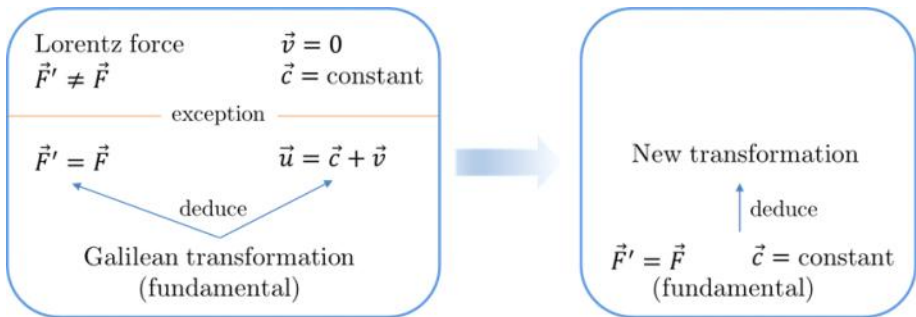
$$\Delta\phi = 0 \Rightarrow v = 0$$

The observed speed of light is independent of medium (ether).

✧ Einstein's Postulations

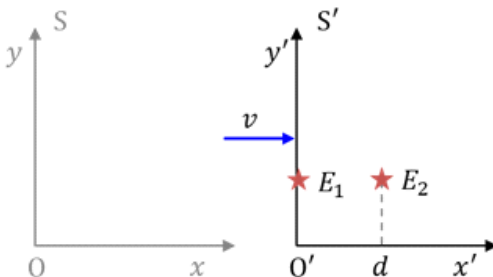
1. The law of physics are invariant in all inertial frames.
2. The speed of light is always c in all inertial frames.

COMMENTS:



D. SIMULTANEITY & SYNCHRONIZING CLOCKS

- (1) In Newtonian thinking, consider two events E_1 and E_2 occur simultaneously in S' frame.



Using Galilean transformation, we obtain

In S' frame:

$$\begin{aligned} x'_1 &= 0, & x'_2 &= d \\ t'_1 &= 0, & t'_2 &= 0 \end{aligned}$$

In S frame:

$$x_1 = vt, \quad x_2 = vt + d$$

$$t_1 = t'_1 = 0, \quad t_2 = t'_2 = 0$$

Since

$$\Delta t' = \Delta t = 0$$

two events are simultaneous in one inertial frame and also simultaneous in any other inertial frame.

(2) Simultaneity and synchronizing clocks

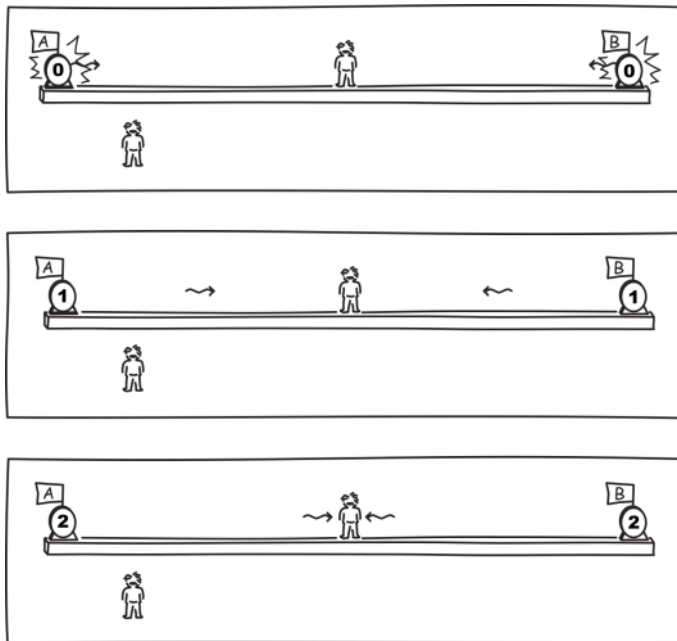
OS:

Einstein for Everyone by John D. Norton

https://sites.pitt.edu/~jdnorton/teaching/HPS_0410/index.html

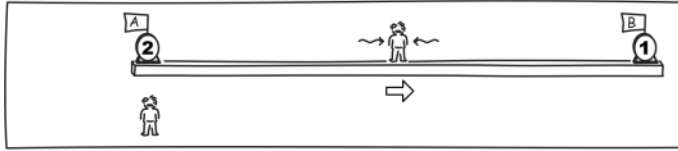
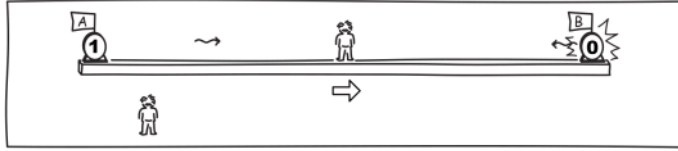
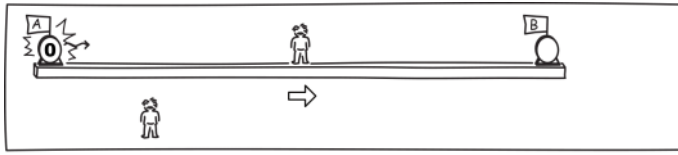
In Einstein thinking, imagine a long platform with an observer located at its midpoint.

1. A-event and B-event flash light simultaneously.



The light arrives the platform observer at the same time, i.e., A-clock and B-clock are properly synchronized.

2. The platform moves with constant velocity and A-event flash the light earlier than B-event.

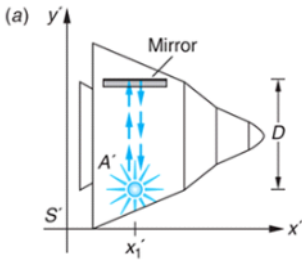


The light arrives the platform observer at the same time, i.e., A-clock is set earlier than the B-clock. A-clock and B-clock are NOT properly synchronized.

Thus, two events that are simultaneous in one inertial frame will not necessarily be simultaneous in any other inertial frame of reference,

$$\Delta t \neq \Delta t'$$

- (3) Consider observer A' and the light source are at rest in the spaceship traveling with velocity v .

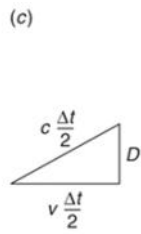
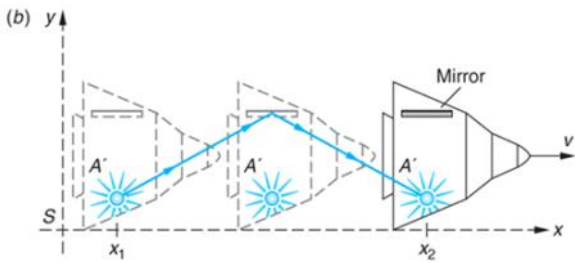


Thus, relative to the light source, the observer A' is in a rest frame (S').

From A' :

$$D = c \frac{\Delta t'}{2}$$





The observer A, at rest on the ground, is in a moving frame (S).
From A:

$$\left(c \frac{\Delta t}{2}\right)^2 = \left(v \frac{\Delta t}{2}\right)^2 + D^2$$

$$(c^2 - v^2) \frac{(\Delta t)^2}{4} = c^2 \frac{(\Delta t')^2}{4}$$

$$(\Delta t)^2 = \frac{c^2}{c^2 - v^2} (\Delta t')^2$$

$$\Delta t = \frac{c \Delta t'}{\sqrt{c^2 - v^2}} = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}}$$

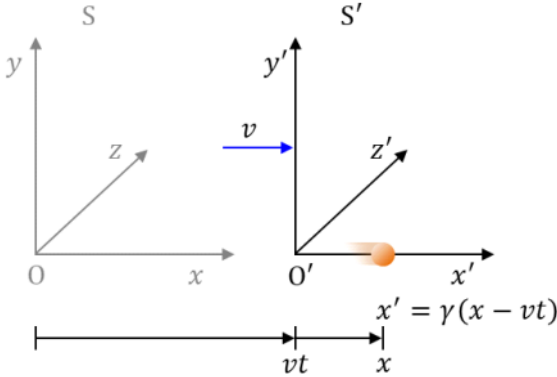
Let $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} > 1$

$$\Delta t = \gamma \Delta t'$$

5-2 Lorentz Transformation

A. LORENTZ TRANSFORMATION

- (1) Derive a new transformation for two inertial frames



$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma(t + \delta)$$

The path of light in S frame

$$x^2 + y^2 + z^2 = c^2 t^2 \Rightarrow c^2 t^2 - x^2 = y^2 + z^2$$

The path of light in S' frame

$$x'^2 + y'^2 + z'^2 = c^2 t'^2 \Rightarrow c^2 t'^2 - x'^2 = y'^2 + z'^2$$

Since

$$y^2 + z^2 = y'^2 + z'^2$$

we obtain

$$\begin{aligned} c^2 t^2 - x^2 &= c^2 t'^2 - x'^2 \\ &= c^2 \gamma^2 (t + \delta)^2 - \gamma^2 (x - vt)^2 \\ &= c^2 \gamma^2 (t^2 + \delta^2 + 2t\delta) - \gamma^2 (x^2 + v^2 t^2 - 2xvt) \\ &= (c^2 \gamma^2 - v^2 \gamma^2) t^2 + c^2 \gamma^2 \delta^2 - \gamma^2 x^2 + \underbrace{c^2 \gamma^2 2t\delta - \gamma^2 2xvt} \end{aligned}$$

Since there is no x and t terms on the L.H.S., we should not have x and t terms on the R.H.S. Thus, we set

$$c^2 \gamma^2 2t\delta - \gamma^2 2xvt = 0 \Rightarrow \delta = -\frac{v}{c^2} x$$

$$\begin{aligned}
 c^2 t^2 - x^2 &= (c^2 \gamma^2 - v^2 \gamma^2) t^2 + c^2 \gamma^2 \left(\frac{v}{c^2} x \right)^2 - \gamma^2 x^2 \\
 &= (c^2 \gamma^2 - v^2 \gamma^2) t^2 + \left(\frac{v^2 \gamma^2}{c^2} - \gamma^2 \right) x^2
 \end{aligned}$$

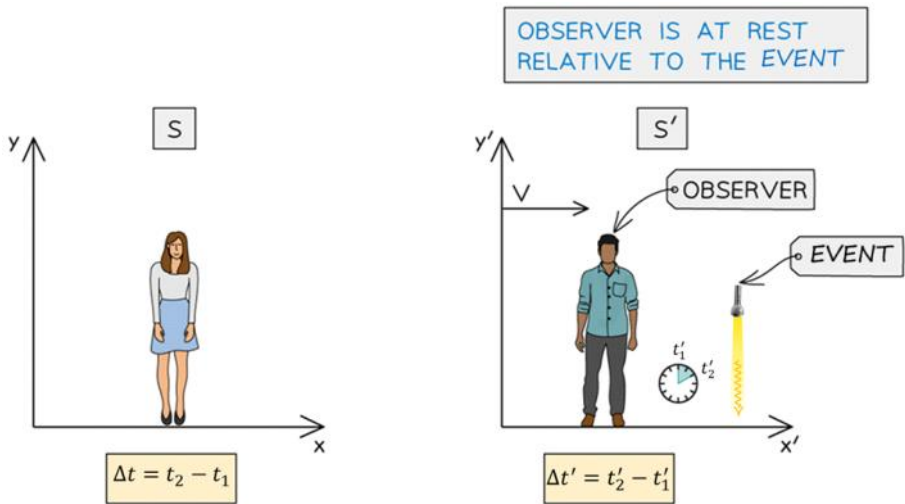
Compare t^2 term (or compare x^2 term, we should obtain the same result):

$$c^2 = c^2 \gamma^2 - v^2 \gamma^2 \Rightarrow \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} > 1$$

Thus, we call the transformation as the Lorentz transformation

$$\begin{aligned}
 x' &= \gamma(x - vt) \\
 y' &= y \\
 z' &= z \\
 t' &= \gamma \left(t - \frac{v}{c^2} x \right)
 \end{aligned}, \text{ where } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

(2) Using Lorentz transformation to derive the time dilation



The time interval measured in S frame is

$$\Delta t = t_2 - t_1$$

Using the inverse Lorentz transformation

$$t = \gamma \left(t' + \frac{v}{c^2} x' \right)$$

we obtain

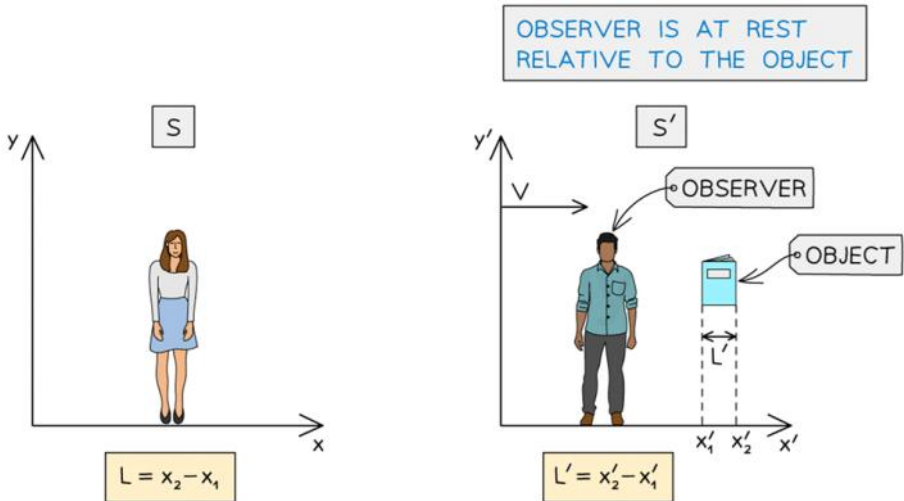
$$\begin{aligned}
 \Delta t &= \gamma \left(t'_2 + \frac{v}{c^2} x'_2 \right) - \gamma \left(t'_1 + \frac{v}{c^2} x'_1 \right) \\
 &= \gamma (t'_2 - t'_1) + \frac{\gamma v}{c^2} (x'_2 - x'_1) \\
 &= \gamma \Delta t' + \frac{\gamma v}{c^2} \Delta x'
 \end{aligned}$$

Since $\Delta x' = 0$, we obtain

$$\Delta t = \gamma \Delta t'$$

Since $\gamma > 1$, the observer in S frame measures a longer time interval than the observer in S' frame.

(3) Using Lorentz transformation to predict the length contraction



The length measured in S' frame is

$$L' = x'_2 - x'_1$$

Using Lorentz transformation

$$x' = \gamma(x - vt)$$

we obtain

$$\begin{aligned}
 L' &= \gamma(x_2 - vt_2) - \gamma(x_1 - vt_1) \\
 &= \gamma(x_2 - x_1) - \gamma v(t_2 - t_1) \\
 &= \gamma L - \gamma v \Delta t
 \end{aligned}$$

Since $\Delta t = 0$

$$L' = \gamma L \Rightarrow L = \frac{L'}{\gamma}$$

Since $\gamma > 1$, the observer in S frame measures a shorter length than the observer in S' frame.

EXAMPLES:

1. Muon decay in experiments:

The decay time $\Delta t' \approx 2 \mu s$

The travel distance $L' = v\Delta t' \approx 600 \text{ m}$

The high above the sea level of muon appears?

ANSWER:

The speed of μ particle in the cosmic ray is $v = 0.998c$

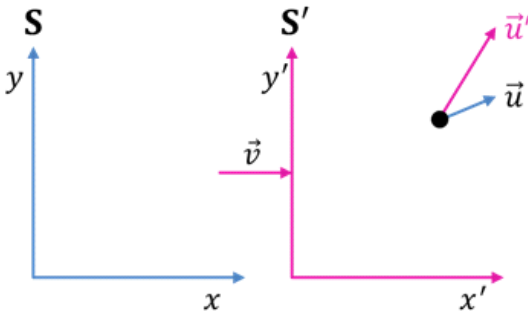
The decay time at earth is

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}} = 30 \mu s$$

$$h = 0.998c \times 30 = 9000 \text{ m}$$

B. RELATIVISTIC ADDITION OF VELOCITIES

- (1) Using Lorentz transformation to derive the addition of velocities



$$u'_x = \frac{dx'}{dt'} = \frac{dx'}{dt} \frac{dt}{dt'}$$

$$u'_y = \frac{dy'}{dt'} = \frac{dy'}{dt} \frac{dt}{dt'}$$

$$u'_z = \frac{dz'}{dt'} = \frac{dz'}{dt} \frac{dt}{dt'}$$

Since

$$x' = \gamma(x - vt) \Rightarrow \frac{dx'}{dt} = \gamma \left(\frac{dx}{dt} - v \right) = \gamma(u_x - v)$$

$$t' = \gamma \left(t - \frac{v}{c^2} x \right) \Rightarrow \frac{dt'}{dt} = \gamma \left(1 - \frac{v}{c^2} \frac{dx}{dt} \right) = \gamma \left(1 - \frac{u_x v}{c^2} \right)$$

thus, we obtain

$$u'_x = \frac{\gamma(u_x - v)}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$$u'_y = \frac{\frac{dy}{dt}}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

$$u'_z = \frac{\frac{dz}{dt}}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

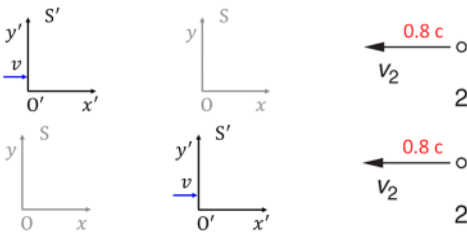
EXAMPLES:

- Find the relative speed of 2 with respect to 1 and the relative speed of 1 with respect to 2.



ANSWER:

- $S = \text{earth}, S' = 1$

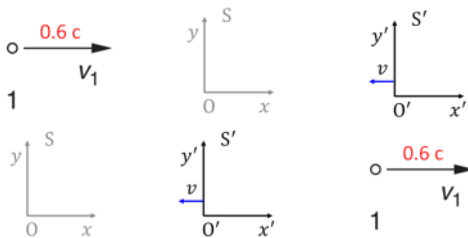


$$u_x = -0.8c$$

$$v = 0.6c$$

$$u'_x = \frac{-0.8c - 0.6c}{1 - \frac{1}{c^2}(-0.8c)(0.6c)} = -0.95c$$

- $S = \text{earth}, S' = 2$



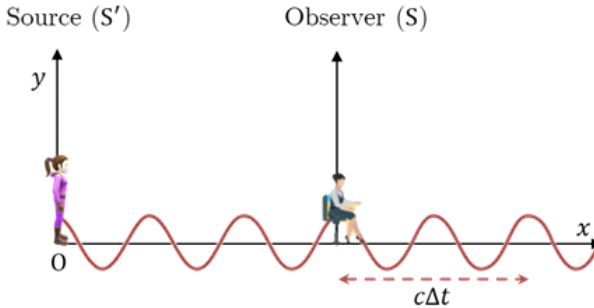
$$u_x = 0.6c$$

$$v = -0.8c$$

$$u'_x = \frac{0.6c + 0.8c}{1 - \frac{1}{c^2}(0.6c)(-0.8c)} = 0.95c$$

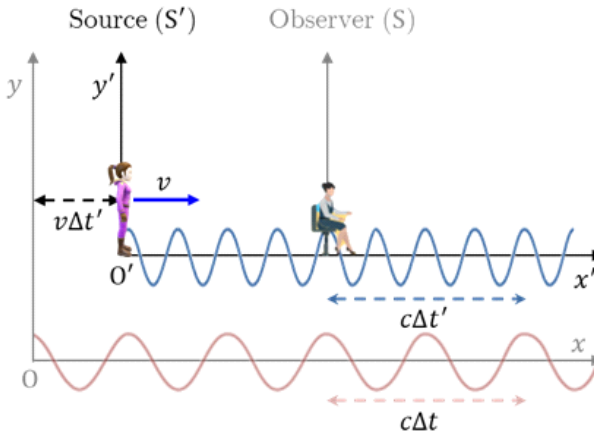
2. Relativistic Doppler effect provides a direct evidence to exhibit the validation of Lorentz transformation.

ANSWER:



$$N\lambda = c\Delta t \Rightarrow f = \frac{c}{\lambda} = \frac{N}{\Delta t}$$

Considering the source approaches observer



$$N\lambda' + v\Delta t' = c\Delta t' \Rightarrow \lambda' = \frac{c\Delta t' - v\Delta t'}{N} = \frac{c - v}{N} \Delta t'$$

$$f' = \frac{c}{\lambda'} = \frac{c}{\frac{(c - v)}{N} \Delta t'} = \frac{c}{c - v} \frac{N}{\Delta t'} = \frac{c}{c - v} \frac{f \Delta t'}{\Delta t}$$

Applying time dilation: $\Delta t = \gamma \Delta t'$, we obtain

$$\begin{aligned}
 f' &= \frac{c}{c-v} \frac{f}{\gamma} \\
 &= \frac{1}{1-\frac{v}{c}} \frac{f}{\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}} \\
 &= \frac{1}{1-\frac{v}{c}} \sqrt{\left(1+\frac{v}{c}\right)\left(1-\frac{v}{c}\right)} f \\
 &= \sqrt{\frac{1+v/c}{1-v/c}} f
 \end{aligned}$$

3. A continuously emitted electromagnetic wave reflected back from a mirror with speed v .
- What is the reflected frequency?
 - If $v \ll c$, we assume $f + f_0 \approx 2f_0$, what is the beat frequency?
 - If beat frequency measurement is accurate to $\pm 5 \text{ Hz}$, how accurate is the v measured?

ANSWER:

(a)

$$\begin{aligned}
 f' &= \sqrt{\frac{c+v}{c-v}} f_0 \\
 f &= \sqrt{\frac{c+v}{c-v}} f' \\
 f &= \frac{c+v}{c-v} f_0
 \end{aligned}$$

(b)

$$\begin{aligned}
 f(c-v) &= f_0(c+v) \\
 (f-f_0)c &= (f+f_0)v \approx 2f_0v \\
 f-f_0 &= \frac{2f_0v}{c} = \frac{2v}{\lambda}
 \end{aligned}$$

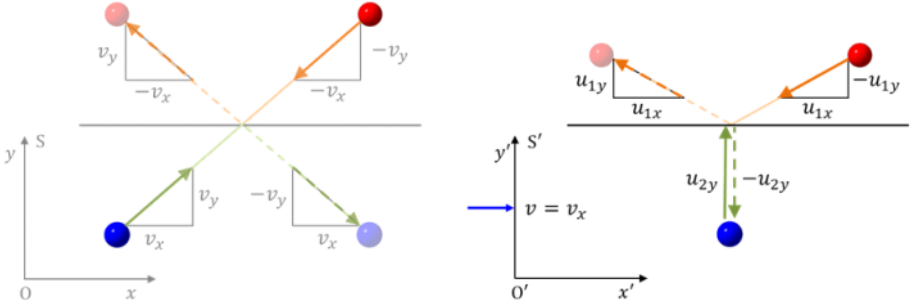
(c)

$$\Delta v = \frac{\Delta(f-f_0)\lambda}{2} = 0.075 \text{ m/s}$$

5-3 Relativistic Momentum and Energy

A. CONSERVATION OF MOMENTUM

- (1) Conservation of momentum arises basically due to the homogeneity and isotropic nature of space. In Lorentz transformation, we have assumed space to be homogeneous and isotropic. Thus, conservation of momentum should be valid in special relativity as well.
- (2) Consider an elastic collision between two equal mass particles from the point of view of two inertial frames S and S'.



In S frame, the center of mass is at rest and thus both particles approach each other with equal and opposite velocities.

$$\vec{p}_1 = (-mv_x, -mv_y)$$

$$\vec{p}_2 = (mv_x, mv_y)$$

$$\vec{p}'_1 = (-mv_x, mv_y)$$

$$\vec{p}'_2 = (mv_x, -mv_y)$$

Conservation of momentum is given as

$$\vec{p}_1 + \vec{p}_2 = \vec{p}'_1 + \vec{p}'_2 = (0,0)$$

- (3) Now consider a reference frame S' moving with velocity $\vec{u} = v_x \hat{x} = v \hat{x}$ with respect to S. The relativistic velocity in S' in terms of those in S as follows.

$$u_{1x} = \frac{-v_x - v}{1 - (-v_x)v/c^2} = -\frac{2v}{1 + v^2/c^2}$$

$$u_{1y} = \frac{v_y}{\gamma(1 - (-v_x)v/c^2)} = \frac{v_y}{1 + v^2/c^2} \sqrt{1 - v^2/c^2}$$

$$u_{2x} = \frac{v_x - v}{1 - v_x v / c^2} = \frac{v - v}{1 - v^2 / c^2} = 0$$

$$u_{2y} = \frac{v_y}{\gamma(1 - v_x v / c^2)} = \frac{v_y}{\sqrt{1 - v^2 / c^2}}$$

Using Newtonian definition of momentum $\vec{p} = m\vec{v}$, then we have

$$\vec{p}_1 = (mu_{1x}, -mu_{1y}) = \left(-\frac{2mv}{1 + v^2 / c^2}, -\frac{mv_y \sqrt{1 - v^2 / c^2}}{1 + v^2 / c^2} \right)$$

$$\vec{p}_2 = (0, mu_{2y}) = \left(0, \frac{mv_y}{\sqrt{1 - v^2 / c^2}} \right)$$

$$\vec{p}'_1 = (-mu_{1x}, mu_{1y}) = \left(-\frac{2mv}{1 + v^2 / c^2}, \frac{mv_y \sqrt{1 - v^2 / c^2}}{1 + v^2 / c^2} \right)$$

$$\vec{p}'_2 = (0, -mu_{2y}) = \left(0, -\frac{mv_y}{\sqrt{1 - v^2 / c^2}} \right)$$

$$\vec{p}_1 + \vec{p}_2 = \left(-\frac{2mv}{1 + v^2 / c^2}, \frac{mv_y}{\sqrt{1 - v^2 / c^2}} - \frac{mv_y \sqrt{1 - v^2 / c^2}}{1 + v^2 / c^2} \right)$$

$$\vec{p}'_1 + \vec{p}'_2 = \left(-\frac{2mv}{1 + v^2 / c^2}, \frac{mv_y \sqrt{1 - v^2 / c^2}}{1 + v^2 / c^2} - \frac{mv_y}{\sqrt{1 - v^2 / c^2}} \right)$$

$$\Rightarrow \vec{p}_1 + \vec{p}_2 \neq \vec{p}'_1 + \vec{p}'_2$$

- (4) We should redefine momentum in such a way that conservation of momentum holds in all reference frames moving at constant relative velocities.

Mathematically, considering the case $p_{1y} + p_{2y} = p'_{1y} + p'_{2y}$ in a general form, i.e., assuming $m_1 v_{1y} \neq m_2 v_{2y}$, thus we have

$$\frac{m_2 v_{2y}}{\sqrt{1 - v^2 / c^2}} - \frac{m_1 v_{1y} \sqrt{1 - v^2 / c^2}}{1 + v^2 / c^2} = \frac{m_1 v_{1y} \sqrt{1 - v^2 / c^2}}{1 + v^2 / c^2} - \frac{m_2 v_{2y}}{\sqrt{1 - v^2 / c^2}}$$

$$\Rightarrow \frac{2m_1 v_{1y} \sqrt{1 - v^2 / c^2}}{1 + v^2 / c^2} - \frac{2m_2 v_{2y}}{\sqrt{1 - v^2 / c^2}} = 0$$

$$\Rightarrow \frac{-2}{\sqrt{1 - v^2 / c^2}} \left(m_1 v_{1y} \frac{1 - v^2 / c^2}{1 + v^2 / c^2} - m_2 v_{2y} \right) = 0$$

Thus, we obtain

$$m_1 v_{1y} \frac{1 - v^2 / c^2}{1 + v^2 / c^2} - m_2 v_{2y} = 0 \Rightarrow \frac{m_1 v_{1y}}{m_2 v_{2y}} = \frac{1 + v^2 / c^2}{1 - v^2 / c^2}$$

Since

$$\begin{aligned}
\frac{1}{\sqrt{1 - \frac{u_{1x}^2}{c^2}}} &= \frac{1}{\sqrt{1 - \frac{1}{c^2} \left(\frac{2v}{1 + v^2/c^2} \right)^2}} \\
&= \frac{1}{\sqrt{\frac{(1 + v^2/c^2)^2}{(1 + v^2/c^2)^2} - \frac{4v^2/c^2}{(1 + v^2/c^2)^2}}} \\
&= \frac{1}{\sqrt{\left(\frac{1 - v^2/c^2}{1 + v^2/c^2} \right)^2}} \\
&= \frac{1 + v^2/c^2}{1 - v^2/c^2}
\end{aligned}$$

we obtain

$$m_1 v_{1y} = \frac{m_2 v_{2y}}{\sqrt{1 - u_{1x}^2/c^2}}$$

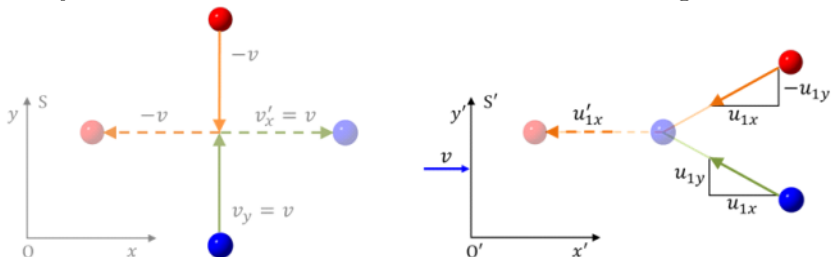
Hence, we use a similar form as above to redefine the momentum as follows:

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1 - \vec{u}^2/c^2}}$$

where $\vec{u}$ is the velocity of the particle and $\vec{p}$ is called the relativistic momentum.

EXAMPLES:

1. Verify the conservation of momentum of the following collision.



ANSWER:

In S frame:

Before collision:

$$\vec{u}_1 = (0, -v), \quad \vec{u}_2 = (0, v)$$

$$\vec{p}_1 = \frac{m\vec{u}_1}{\sqrt{1 - \vec{u}_1^2/c^2}} = \left(0, -\frac{mv}{\sqrt{1 - v^2/c^2}}\right)$$

$$\vec{p}_2 = \frac{m\vec{u}_2}{\sqrt{1 - \vec{u}_2^2/c^2}} = \left(0, \frac{mv}{\sqrt{1 - v^2/c^2}}\right)$$

After collision:

$$\vec{u}'_1 = (-v, 0), \quad \vec{u}'_2 = (v, 0)$$

$$\vec{p}'_1 = \frac{m\vec{u}'_1}{\sqrt{1 - \vec{u}'_1{}^2/c^2}} = \left(-\frac{mv}{\sqrt{1 - v^2/c^2}}, 0\right)$$

$$\vec{p}'_2 = \frac{m\vec{u}'_2}{\sqrt{1 - \vec{u}'_2{}^2/c^2}} = \left(\frac{mv}{\sqrt{1 - v^2/c^2}}, 0\right)$$

$$\Rightarrow \vec{p}_1 + \vec{p}_2 = \vec{p}'_1 + \vec{p}'_2 = (0, 0)$$

In S' frame:

Before collision:

$$u_{1x} = \frac{v_x - v}{1 - v_x v/c^2} = \frac{0 - v}{1 - 0 \cdot v/c^2} = -v$$

$$u_{1y} = \frac{v_y}{\gamma(1 - v_x v/c^2)} = \frac{v}{\gamma(1 - 0 \cdot v/c^2)} = \frac{v}{\gamma}$$

$$\vec{u}_1 = \left(-v, -\frac{v}{\gamma}\right)$$

$$p_{1x} = \frac{mu_{1x}}{\sqrt{1 - \vec{u}_1^2/c^2}}$$

$$= \frac{m(-v)}{\sqrt{1 - \vec{u}_1^2/c^2}}$$

$$= -\frac{mv}{\sqrt{1 - \frac{1}{c^2}\left(v^2 + \frac{v^2}{\gamma^2}\right)}}$$

$$= -\frac{mv}{\sqrt{1 - \frac{v^2}{c^2} - \frac{v^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)}}$$

$$= -\frac{mv}{\sqrt{1 - \frac{2v^2}{c^2} + \frac{v^4}{c^4}}}$$

$$= -\frac{mv}{\sqrt{(1 - v^2/c^2)^2}}$$

$$= -\frac{mv}{1 - v^2/c^2}$$

$$\begin{aligned}
p_{1y} &= \frac{mu_{1y}}{\sqrt{1 - \vec{u}_1^2/c^2}} \\
&= \frac{m\left(-\frac{v}{\gamma}\right)}{\sqrt{1 - \frac{1}{c^2}\left(v^2 + \frac{v^2}{\gamma^2}\right)}} \\
&= -\frac{mv\sqrt{1 - v^2/c^2}}{\sqrt{(1 - v^2/c^2)^2}} \\
&= -\frac{mv}{\sqrt{1 - v^2/c^2}} \\
\vec{p}_1 &= \left(-\frac{mv}{1 - v^2/c^2}, -\frac{mv}{\sqrt{1 - v^2/c^2}}\right) \\
\vec{p}_2 &= \left(-\frac{mv}{1 - v^2/c^2}, \frac{mv}{\sqrt{1 - v^2/c^2}}\right)
\end{aligned}$$

After collision:

$$\begin{aligned}
u'_{1x} &= \frac{v'_x - v}{1 - v'_x v/c^2} = \frac{-v - v}{1 + vv/c^2} = -\frac{2v}{1 + v^2/c^2} \\
u'_{1y} &= \frac{v'_y}{\gamma(1 - v'_x v/c^2)} = \frac{0}{\gamma(1 + vv/c^2)} = 0 \\
\vec{u}'_1 &= \left(-\frac{2v}{1 + v^2/c^2}, 0\right)
\end{aligned}$$

$$\begin{aligned}
p'_{1x} &= \frac{mu'_{1x}}{\sqrt{1 - \vec{u}'_1{}^2/c^2}} \\
&= \frac{m\left(-\frac{2v}{1 + v^2/c^2}\right)}{\sqrt{1 - \frac{1}{c^2}\left(\frac{2v}{1 + v^2/c^2}\right)^2}} \\
&= -\frac{\frac{2mv}{1 + v^2/c^2}}{\sqrt{\frac{(1 + v^2/c^2)^2}{(1 + v^2/c^2)^2} - \frac{4v^2/c^2}{(1 + v^2/c^2)^2}}} \\
&= -\frac{\frac{2mv}{1 + v^2/c^2}}{\sqrt{\left(\frac{1 - v^2/c^2}{1 + v^2/c^2}\right)^2}} \\
&= -\frac{2mv}{1 - v^2/c^2} \\
p'_{1y} &= \frac{mu'_{1y}}{\sqrt{1 - \vec{u}'_1{}^2/c^2}} = 0 \\
\vec{p}'_1 &= \left(-\frac{2mv}{1 - v^2/c^2}, 0\right) \\
\vec{p}'_2 &= (0,0) \\
\Rightarrow \vec{p}_1 + \vec{p}_2 &= \vec{p}'_1 + \vec{p}'_2 = \left(-\frac{2mv}{1 - v^2/c^2}, 0\right)
\end{aligned}$$

B. CONSERVATION OF ENERGY

- (1) The work-energy theorem

The kinetic energy E_k is the work done by a net force in accelerating a particle from rest to velocity v ,

$$E_k = \int \vec{F} \cdot d\vec{r}$$

Since

$$\begin{aligned}
\vec{F} &= \frac{d\vec{p}}{dt} \\
&= \frac{d}{dt} \left(\frac{m\vec{u}}{\sqrt{1-u^2/c^2}} \right) \\
&= m \left[\frac{1}{\sqrt{1-u^2/c^2}} + \frac{u^2/c^2}{(1-u^2/c^2)^{3/2}} \right] \frac{d\vec{u}}{dt} \\
&= \frac{m \frac{d\vec{u}}{dt}}{(1-u^2/c^2)^{3/2}}
\end{aligned}$$

we obtain

$$\begin{aligned}
E_k &= \int \frac{m \frac{d\vec{u}}{dt}}{(1-u^2/c^2)^{3/2}} \cdot d\vec{r} \\
&= m \int \frac{d\vec{u} \cdot \vec{u}}{(1-u^2/c^2)^{3/2}} \\
&= m \int_0^v \frac{\frac{1}{2} du^2}{(1-u^2/c^2)^{3/2}} \\
&= \frac{mc^2}{\sqrt{1-u^2/c^2}} - mc^2 \\
\Rightarrow E_k + mc^2 &= \frac{mc^2}{\sqrt{1-u^2/c^2}}
\end{aligned}$$

Thus, we can define the relativistic energy

$$E = \frac{mc^2}{\sqrt{1-u^2/c^2}} = \underbrace{E_k}_{\text{kinetic energy}} + \underbrace{mc^2}_{\text{rest energy}}$$

where $\vec{u}$ is the velocity of the particle. mc^2 is called the rest energy because there is no $\vec{u}$ in the formula.

(2) Taking the square of both sides

$$\frac{m^2 c^4}{1-u^2/c^2} = (E_k + mc^2)^2 = E_k^2 + 2E_k mc^2 + m^2 c^4$$

$$\begin{aligned}
 \Rightarrow E_k^2 + 2E_k mc^2 &= \frac{m^2 c^4}{1 - u^2/c^2} - m^2 c^4 \\
 &= m^2 c^4 \frac{u^2/c^2}{1 - u^2/c^2} \\
 &= \frac{m^2 u^2 c^2}{1 - u^2/c^2} \\
 &= p^2 c^2
 \end{aligned}$$

Thus, the square of the relativistic energy

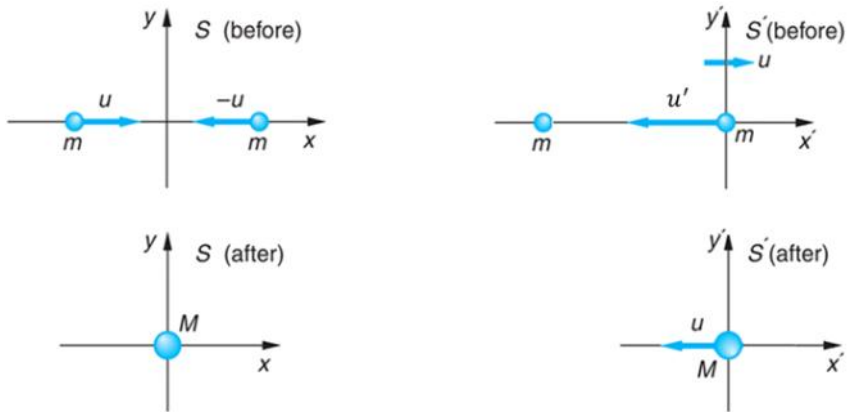
$$E^2 = \underbrace{E_k^2 + 2E_k mc^2}_{=p^2 c^2} + m^2 c^4 = p^2 c^2 + m^2 c^4$$

We then obtain

$$E = \sqrt{p^2 c^2 + m^2 c^4}$$

EXAMPLES:

1. Verify the conservation of energy of a relativistic inelastic collision.



ANSWER:

- In S frame:

Before collision:

$$E_b = \frac{mc^2}{\sqrt{1 - u^2/c^2}} + \frac{mc^2}{\sqrt{1 - (-u)^2/c^2}} = \frac{2mc^2}{\sqrt{1 - u^2/c^2}}$$

After collision:

$$E_a = Mc^2 \dots \text{rest energy}$$

Energy will be conserved in S if $E_b = E_a$, i.e., if

$$Mc^2 = \frac{2mc^2}{\sqrt{1 - u^2/c^2}}$$

- In S' frame:

Before collision:

$$\begin{aligned}
 u' &= \frac{-u - u}{1 - (-u)u/c^2} = \frac{-2u}{1 + u^2/c^2} \\
 E'_b &= \underbrace{mc^2}_{\text{rest energy}} + \frac{mc^2}{\sqrt{1 - \frac{1}{c^2} \left(\frac{-2u}{1 + u^2/c^2} \right)^2}} \\
 &= mc^2 + \frac{mc^2}{\sqrt{\left(\frac{1 - u^2/c^2}{1 + u^2/c^2} \right)^2}} \\
 &= mc^2 \left(1 + \frac{1 + u^2/c^2}{1 - u^2/c^2} \right) \\
 &= \frac{2mc^2}{1 - u^2/c^2}
 \end{aligned}$$

After collision:

$$E'_a = \frac{Mc^2}{\sqrt{1 - (-u)^2/c^2}} = \frac{Mc^2}{\sqrt{1 - u^2/c^2}}$$

The energy will be conserved in S' if $E'_b = E'_a$, i.e., if

$$Mc^2 = \frac{2mc^2}{\sqrt{1 - u^2/c^2}}$$

Therefore, the conservation of energy will hold in all inertial frames.

2. Show that if momentum is conserved in any reference frame relative to S , the loss of the kinetic energy of the system is given by

$$E_k = Mc^2 - 2mc^2$$

ANSWER:

In S' frame:

$$u' = \frac{-u - u}{1 - (-u)u/c^2} = \frac{-2u}{1 + u^2/c^2}$$

$$p = \frac{mu'}{\sqrt{1 - u'^2/c^2}} = \frac{m \left(\frac{2u}{1 + u^2/c^2} \right)}{\sqrt{1 - \frac{1}{c^2} \left(\frac{-2u}{1 + u^2/c^2} \right)^2}} = \frac{2mu}{1 - u^2/c^2}$$

$$p' = \frac{Mu}{\sqrt{1 - u/c^2}}$$

Conservation of momentum requires that $p = p'$, or

$$\begin{aligned} \frac{Mu}{\sqrt{1 - u/c^2}} &= \frac{2mu}{1 - u^2/c^2} \\ \Rightarrow Mc^2 &= \frac{2mc^2}{\sqrt{1 - u/c^2}} = E_k + 2mc^2 \\ \Rightarrow E_k &= Mc^2 - 2mc^2 \end{aligned}$$

3. A stationary body explodes into two fragments each of mass 1.0 kg that move apart at speeds of 0.6c relative to the original body. Find the mass of the original body.

ANSWER:

The rest energy of the original body must equal the sum of the total energies of the fragments.

$$\begin{aligned} \frac{m_1 c^2}{\sqrt{1 - u_1^2/c^2}} + \frac{m_2 c^2}{\sqrt{1 - u_2^2/c^2}} &= mc^2 \\ m &= \frac{1.0}{\sqrt{1 - (0.6)^2}} + \frac{1.0}{\sqrt{1 - (0.6)^2}} = \frac{2}{\sqrt{1 - (0.6)^2}} = 2.5 \text{ kg} \end{aligned}$$

5-4 Lorentz Invariance

A. MINKOWSKI SPACE

- (1) Invariant under Galilean transformation

The length:

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

Since

$$\begin{aligned}(\Delta s')^2 &= (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 \\&= (\Delta x - v\Delta t)^2 + (\Delta y)^2 + (\Delta z)^2 \\&= (\Delta x)^2 - 2v\Delta x\Delta t + v^2(\Delta t)^2 + (\Delta y)^2 + (\Delta z)^2 \\&= (\Delta s)^2 - v^2(\Delta t)^2\end{aligned}$$

Let $\Delta t = 0$ and we obtain

$$\Delta s' = \Delta s$$

Define a vector in Euclidean space:

$$\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \mathbf{X}^\dagger \cdot \mathbf{X} = (\Delta s)^2$$

- (2) Invariant under the Lorentz transformation

The spacetime interval:

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

Since

$$\begin{aligned}(\Delta s')^2 &= (c\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 \\&= c^2\gamma^2\left(\Delta t - \frac{v}{c^2}\Delta x\right)^2 - \gamma^2(\Delta x - v\Delta t)^2 - (\Delta y)^2 - (\Delta z)^2 \\&= \gamma^2\left(c^2(\Delta t)^2 - 2v\Delta t\Delta x + \frac{v^2}{c^2}(\Delta x)^2\right) \\&\quad - \gamma^2((\Delta x)^2 - 2v\Delta x\Delta t + v^2(\Delta t)^2) - (\Delta y)^2 - (\Delta z)^2 \\&= \gamma^2(\Delta t)^2(c^2 - v^2) - \gamma^2\Delta x^2\left(1 - \frac{v^2}{c^2}\right) - (\Delta y)^2 - (\Delta z)^2 \\&= c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \\&= (\Delta s)^2\end{aligned}$$

$(\Delta s)^2$ is invariant under the Lorentz transformation.

Define a four-vector of spacetime:

$$\mathbf{X} = \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} \Rightarrow \mathbf{X}^\dagger \cdot \mathbf{X} \neq (\Delta s)^2$$

We introduce a metric matrix $\boldsymbol{\eta}$ as,

$$\boldsymbol{\eta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \dots \text{Minkowski metric matrix}$$

and obtain

$$\mathbf{X}^\dagger \cdot \boldsymbol{\eta} \cdot \mathbf{X} = c^2 t^2 - x^2 - y^2 - z^2 = (\Delta s)^2$$

(3) Lorentz transformation in a symmetric matrix form

$$\begin{aligned} x' &= \gamma \left(x - \frac{v}{c}(ct) \right) \\ y' &= y \\ z' &= z \\ ct' &= \gamma \left((ct) - \frac{v}{c}x \right) \end{aligned} \Rightarrow \begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma \frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma \frac{v}{c} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}$$

Let

$$\boldsymbol{\Lambda} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma \frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma \frac{v}{c} & 0 & 0 & \gamma \end{pmatrix}$$

The Lorentz transformation is

$$\mathbf{X}' = \boldsymbol{\Lambda} \cdot \mathbf{X}$$

Since

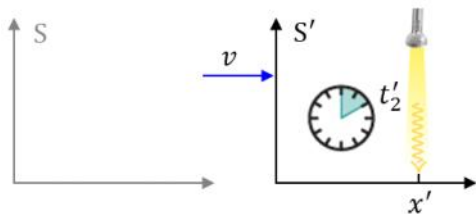
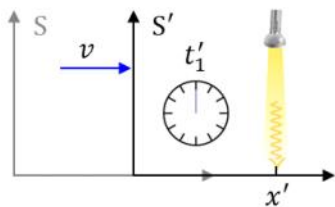
$$\begin{aligned} \underbrace{\mathbf{X}'^\dagger \cdot \boldsymbol{\eta} \cdot \mathbf{X}'}_{=(\Delta s')^2} &= \mathbf{X}^\dagger \cdot \boldsymbol{\Lambda}^\dagger \cdot \boldsymbol{\eta} \cdot \boldsymbol{\Lambda} \cdot \mathbf{X} = \underbrace{\mathbf{X}^\dagger \cdot \boldsymbol{\eta} \cdot \mathbf{X}}_{=(\Delta s)^2} \\ &\Rightarrow \boldsymbol{\Lambda}^\dagger \cdot \boldsymbol{\eta} \cdot \boldsymbol{\Lambda} = \boldsymbol{\eta} \end{aligned}$$

EXAMPLES:

1. Using spacetime interval to derive the time dilation and length contraction.

ANSWER:

(a)



$$(\Delta s')^2 = (c\Delta t')^2$$

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2$$

Since $(\Delta s')^2 = (\Delta s)^2$

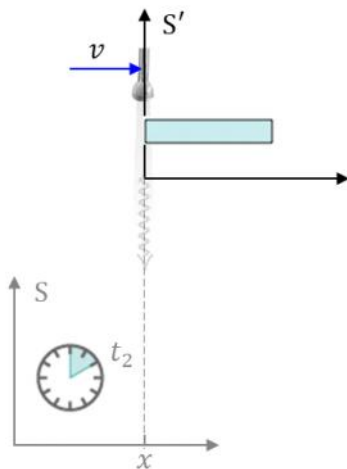
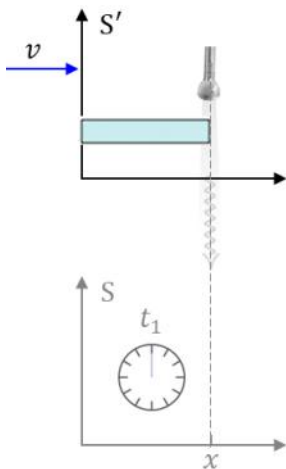
$$(c\Delta t)^2 - (\Delta x)^2 = (c\Delta t')^2$$

$$(c\Delta t)^2 - (v\Delta t)^2 = (c\Delta t')^2$$

$$(\Delta t)^2 = \frac{c^2}{c^2 - v^2} (\Delta t')^2$$

$$\Delta t = \sqrt{\frac{c^2}{c^2 - v^2}} \Delta t' = \frac{1}{\sqrt{1 - v^2/c^2}} \Delta t' = \gamma \Delta t'$$

(b) Length contraction



$$(\Delta s')^2 = (c\Delta t')^2 - (\Delta x')^2 = c^2 \left(\frac{L'}{v} \right)^2 - L'^2$$

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 = c^2 \left(\frac{L}{v} \right)^2$$

$$\text{Since } (\Delta s')^2 = (\Delta s)^2$$

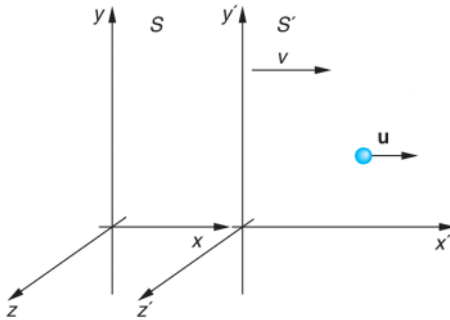
$$c^2 \frac{L'^2}{v^2} - L'^2 = c^2 \frac{L^2}{v^2}$$

$$L'^2 = \frac{c^2/v^2}{c^2/v^2 - 1} L^2 = \frac{1}{1 - v^2/c^2} L^2$$

$$L' = \gamma L \Rightarrow L = \frac{L'}{\gamma}$$

B. THE REST ENERGY & THE REST MASS

- (1) The relationship between energy and momentum in two inertial frames



In S frame:

$$p_x = \frac{mu}{\sqrt{1 - u^2/c^2}}, \quad p_y, \quad p_z, \quad E = \frac{mc^2}{\sqrt{1 - u^2/c^2}}$$

In S' frame:

$$\begin{aligned}
p'_x &= \frac{mu'}{\sqrt{1 - u'^2/c^2}} \\
&= \frac{m\left(\frac{u-v}{1-uv/c^2}\right)}{\sqrt{1 - \frac{1}{c^2}\left(\frac{u-v}{1-uv/c^2}\right)^2}} \\
&= \frac{m\left(\frac{u-v}{1-uv/c^2}\right)}{\sqrt{\frac{(1-uv/c^2)^2}{(1-uv/c^2)^2} - \frac{(u-v)^2/c^2}{(1-uv/c^2)^2}}} \\
&= \frac{m\left(\frac{u-v}{1-uv/c^2}\right)}{\sqrt{\frac{1 - \frac{v^2}{c^2} - \frac{u^2}{c^2} + \frac{u^2v^2}{c^4}}{(1-uv/c^2)^2}}} \\
&= \frac{m(u-v)}{\sqrt{(1-v^2/c^2)(1-u^2/c^2)}} \\
&= \frac{1}{\sqrt{1-v^2/c^2}} \left(\frac{mu}{\sqrt{1-u^2/c^2}} - \frac{mv}{\sqrt{1-u^2/c^2}} \right) \\
&= \gamma \left(p_x - \frac{v}{c^2} E \right) \\
p'_y &= p_y \\
p'_z &= p_z
\end{aligned}$$

$$\begin{aligned}
E' &= \frac{mc^2}{\sqrt{1 - u'^2/c^2}} \\
&= \frac{mc^2}{\sqrt{1 - \frac{1}{c^2} \left(\frac{u - v}{1 - uv/c^2} \right)^2}} \\
&= \frac{mc^2}{\sqrt{\frac{(1 - uv/c^2)^2}{(1 - uv/c^2)^2} - \frac{(u - v)^2/c^2}{(1 - uv/c^2)^2}}} \\
&= \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2} - \frac{u^2}{c^2} + \frac{u^2 v^2}{c^4}}} \\
&= \frac{mc^2(1 - uv/c^2)}{\sqrt{(1 - v^2/c^2)(1 - u^2/c^2)}} \\
&= \frac{1}{\sqrt{1 - v^2/c^2}} \left(\frac{mc^2}{\sqrt{1 - u^2/c^2}} - v \frac{mu}{\sqrt{1 - u^2/c^2}} \right) \\
&= \gamma(E - vp_x)
\end{aligned}$$

In symmetric matrix form:

$$\begin{pmatrix} p'_x \\ p'_y \\ p'_z \\ E'/c \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma \frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma \frac{v}{c} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix}$$

Define a four-vector of momentum-energy:

$$\mathbf{P} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix}$$

We obtain the Lorentz transformation

$$\mathbf{P}' = \mathbf{\Lambda} \cdot \mathbf{P}$$

(2) The rest energy

Using Minkowski metric $\boldsymbol{\eta}$, we obtain

$$\begin{aligned}
\mathbf{P}^\dagger \cdot \boldsymbol{\eta} \cdot \mathbf{P} &= E^2 - (p_x c)^2 - (p_y c)^2 - (p_z c)^2 \\
&= \frac{(mc^2)^2}{1 - u^2/c^2} - \frac{(mu_x c)^2}{1 - u^2/c^2} - \frac{(mu_y c)^2}{1 - u^2/c^2} - \frac{(mu_z c)^2}{1 - u^2/c^2} \\
&= \left[\frac{1 - (u_x^2 + u_y^2 + u_z^2)/c^2}{1 - u^2/c^2} \right] (mc^2)^2 \\
&= (mc^2)^2
\end{aligned}$$

Since

$$\begin{aligned}
(m'c^2)^2 &= E'^2 - (p'_x c)^2 - (p'_y c)^2 - (p'_z c)^2 \\
&= \gamma^2 (E - vp_x)^2 - \left(\gamma \left(p_x - \frac{v}{c^2} E \right) c \right)^2 - (p_y c)^2 - (p_z c)^2 \\
&= \gamma^2 \left(E^2 - 2Ev p_x + v^2 p_x^2 - p_x^2 c^2 + 2p_x v E - \frac{v^2}{c^2} E^2 \right) \\
&\quad - (p_y c)^2 - (p_z c)^2 \\
&= \gamma^2 \left[\left(1 - \frac{v^2}{c^2} \right) E^2 - (c^2 - v^2) p_x^2 \right] - (p_y c)^2 - (p_z c)^2 \\
&= E^2 - \frac{c^2 - v^2}{1 - v^2/c^2} p_x^2 - (p_y c)^2 - (p_z c)^2 \\
&= E^2 - (p_x c)^2 - (p_y c)^2 - (p_z c)^2 \\
&= (mc^2)^2
\end{aligned}$$

$(mc^2)^2$ is invariant under a Lorentz transformation.

(3) The rest mass

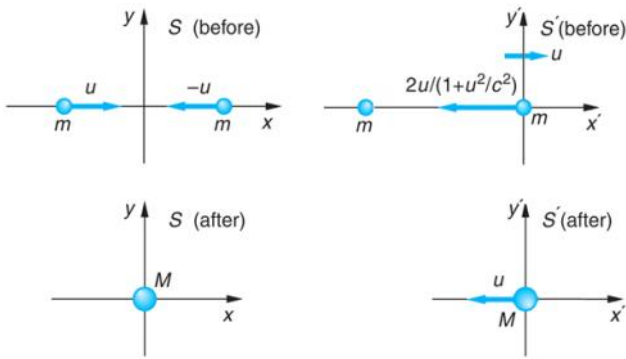
Since

$$(mc^2)^2 = E^2 - (p_x c)^2 - (p_y c)^2 - (p_z c)^2$$

we obtain

$$m = \frac{\sqrt{E^2 - (p_x c)^2 - (p_y c)^2 - (p_z c)^2}}{c^2}$$

Considering an atom composites two identical moving particles



In S frame (center-of-mass):

$$p_1 = \frac{m_1 u_1}{\sqrt{1 - u^2/c^2}} = \frac{mu}{\sqrt{1 - u^2/c^2}}$$

$$p_2 = \frac{m_2 u_2}{\sqrt{1 - u^2/c^2}} = -\frac{mu}{\sqrt{1 - u^2/c^2}}$$

The total momentum is $p = p_1 + p_2 = 0$

$$E_1 = \sqrt{(m_1 c^2)^2 + (p_1 c)^2} = \sqrt{m^2 c^4 + \frac{m^2 u^2 c^2}{1 - u^2/c^2}} = \frac{mc^2}{\sqrt{1 - u^2/c^2}}$$

$$E_2 = \sqrt{(m_2 c^2)^2 + (p_2 c)^2} = \sqrt{m^2 c^4 + \frac{m^2 u^2 c^2}{1 - u^2/c^2}} = \frac{mc^2}{\sqrt{1 - u^2/c^2}}$$

The total energy is

$$E = E_1 + E_2 = \frac{2mc^2}{\sqrt{1 - u^2/c^2}}$$

The rest mass is

$$M = \frac{\sqrt{E^2 - p^2 c^2}}{c^2} = \frac{E}{c^2} = \frac{2m}{\sqrt{1 - u^2/c^2}}$$

In S' frame (Lab frame): $\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$

$$p'_1 = 0$$

$$p'_2 = \gamma \left(p_2 - \frac{u}{c^2} E_2 \right)$$

$$= \gamma \left(-\frac{mu}{\sqrt{1 - u^2/c^2}} - \frac{u}{c^2} \frac{mc^2}{\sqrt{1 - u^2/c^2}} \right)$$

$$= -\frac{2mu}{1 - u^2/c^2}$$

The total momentum is

$$p' = p'_1 + p'_2 = -\frac{2mu}{1 - u^2/c^2}$$

Since

$$\begin{aligned} E'_1 &= \gamma(E_1 - up_1) \\ &= \gamma\left(\frac{mc^2}{\sqrt{1 - u^2/c^2}} - \frac{mu}{\sqrt{1 - u^2/c^2}}u\right) \\ &= \frac{mc^2}{1 - u^2/c^2}\left(1 - \frac{u^2}{c^2}\right) \\ &= mc^2 \\ E'_2 &= \gamma(E_2 - up_2) \\ &= \gamma\left(\frac{mc^2}{\sqrt{1 - u^2/c^2}} + \frac{mu}{\sqrt{1 - u^2/c^2}}u\right) \\ &= \left(\frac{1 + u^2/c^2}{1 - u^2/c^2}\right)mc^2 \end{aligned}$$

The total energy is

$$E = E'_1 + E'_2 = mc^2 + \left(\frac{1 + u^2/c^2}{1 - u^2/c^2}\right)mc^2 = \frac{2mc^2}{1 - u^2/c^2}$$

The rest mass is

$$\begin{aligned} M &= \frac{\sqrt{E^2 - p^2c^2}}{c^2} \\ &= \frac{\sqrt{\left(\frac{2mc^2}{1 - u^2/c^2}\right)^2 - \left(-\frac{2mu}{1 - u^2/c^2}\right)^2}c^2}{c^2} \\ &= \frac{2mc^2}{1 - u^2/c^2} \frac{\sqrt{1 - u^2/c^2}}{c^2} \\ &= \frac{2m}{\sqrt{1 - u^2/c^2}} \end{aligned}$$

OS:

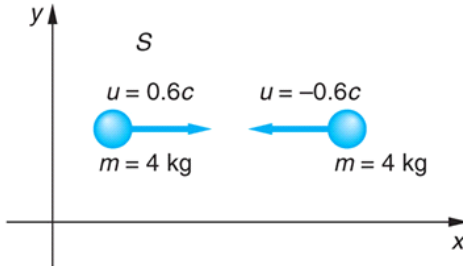
From the relativistic energy

$$E'_1 = \sqrt{(mc^2)^2 + (p'_1c)^2} = mc^2$$

$$\begin{aligned}
 E'_2 &= \sqrt{(mc^2)^2 + (p'_2 c)^2} \\
 &= \sqrt{m^2 c^4 + 4\gamma^4 m^2 u^2 c^2} \\
 &= mc^2 \sqrt{\frac{1 - 2\frac{u^2}{c^2} + \frac{u^4}{c^4} + 4\frac{u^2}{c^2}}{(1 - u^2/c^2)^2}} \\
 &= \left(\frac{1 + u^2/c^2}{1 - u^2/c^2}\right) mc^2
 \end{aligned}$$

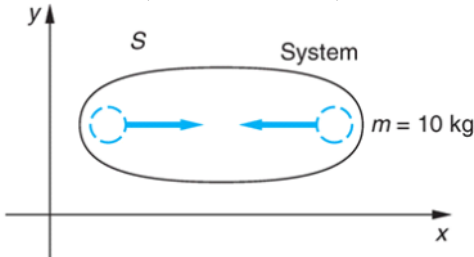
EXAMPLES:

1. Considering an atom composites two identical moving particles. The rest mass of each particle is **4 kg**. What is the rest mass of the atom?



ANSWER:

- In S frame (center-of-mass):



$$p_{1x} = \frac{4 \times 0.6c}{\sqrt{1 - \frac{(0.6c)^2}{c^2}}} = 3c$$

$$p_{2x} = -3c$$

$$\text{Total momentum: } p = p_{1x} + p_{2x} = 0$$

$$E_1 = \sqrt{(mc^2)^2 + (pc)^2} = \sqrt{4^2 + 3^2}c^2 = 5c^2$$

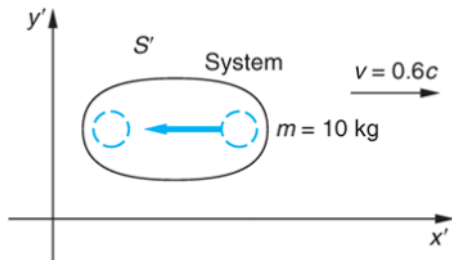
$$E_2 = 5c^2$$

Total energy: $E = E_1 + E_2 = 10c^2$

The rest mass of the atom is

$$M = \frac{\sqrt{E^2 - p^2 c^2}}{c^2} = \sqrt{10^2 - 0^2} = 10$$

- In S' frame (Lab frame):



$$p'_{1x} = 0$$

$$p'_{2x} = \gamma \left(p_{2x} - v \frac{E_2}{c^2} \right) = \frac{-3c - 0.6 \times 5}{0.8} = -7.5c$$

Total momentum: $p = -7.5c$

$$E'_1 = 4c^2$$

$$E'_2 = \gamma (E_2 - v p_{2x}) = \frac{5c^2 + 0.6c \times 3c}{\sqrt{1 - 0.6^2}} = 8.5c^2$$

Total energy: $E = 4c^2 + 8.5c^2 = 12.5c^2$

The rest mass of the atom is $M = \sqrt{12.5^2 - 7.5^2} = 10$

2. An unstable particle having a mass of 3.34×10^{-27} kg is initially at rest. The particle decays into two fragments that fly off with velocity of $0.987c$ and $-0.868c$. Find the rest masses of the fragments?

ANSWER:

Conservation of energy in CM frame

$$M c^2 = \frac{m_1 c^2}{\sqrt{1 - \frac{u_1^2}{c^2}}} + \frac{m_2 c^2}{\sqrt{1 - \frac{u_2^2}{c^2}}} \Rightarrow M = \frac{m_1}{\sqrt{1 - \frac{u_1^2}{c^2}}} + \frac{m_2}{\sqrt{1 - \frac{u_2^2}{c^2}}}$$

$$\frac{1}{\sqrt{1 - \frac{u_1^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{(0.987c)^2}{c^2}}} = 6.22$$

$$\frac{1}{\sqrt{1 - \frac{u_2^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{(-0.868c)^2}{c^2}}} = 2.01$$

$$\Rightarrow 3.34 \times 10^{-27} = 6.22m_1 + 2.01m_2$$

Conservation of momentum in CM frame

$$0 = \frac{m_1 u_1}{\sqrt{1 - \frac{u_1^2}{c^2}}} - \frac{m_2 u_2}{\sqrt{1 - \frac{u_2^2}{c^2}}}$$

$$\Rightarrow m_1 = \frac{u_2}{u_1} \frac{\sqrt{1 - \frac{u_1^2}{c^2}}}{\sqrt{1 - \frac{u_2^2}{c^2}}} m_2 = \frac{0.868c}{0.987c} \times \frac{2.01}{6.22} m_2 = 0.284m_2$$

$$\Rightarrow 3.34 \times 10^{-27} = 6.22 \times 0.284m_2 + 2.01m_2 = 3.78m_2$$

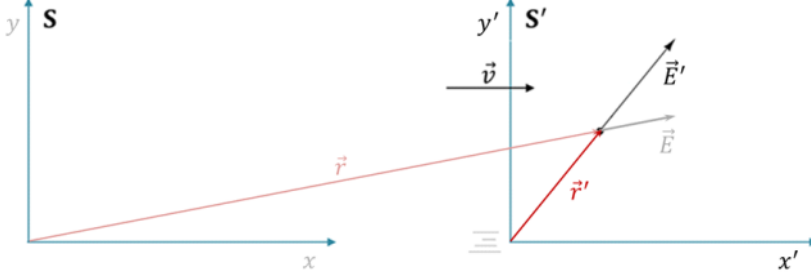
$$\Rightarrow m_2 = 8.84 \times 10^{-28}$$

$$\Rightarrow m_1 = 2.51 \times 10^{-28}$$

5-5 Electric Field Measured in Different Inertial Frames

A. LORENTZ TRANSFORMATION OF ELECTROMAGNETIC FIELDS

- (1) Consider a charged particle moving in constant velocity



We can decompose $\vec{r}$ and $\vec{r}'$ into $\vec{r}_{\parallel}$, $\vec{r}_{\perp}$, $\vec{r}'_{\parallel}$, and $\vec{r}'_{\perp}$ which lie parallel and perpendicular to $\vec{v}$. Thus, we obtain

$$\begin{array}{ll} \text{Lorentz transformation} & \text{inverse Lorentz transformation} \\ \vec{r}'_{\perp} = \vec{r}_{\perp} & \vec{r}_{\perp} = \vec{r}'_{\perp} \\ \vec{r}'_{\parallel} = \gamma(\vec{r}_{\parallel} - \vec{v}t) & \vec{r}_{\parallel} = \gamma(\vec{r}'_{\parallel} + \vec{v}t') \\ t' = \gamma\left(t - \frac{\vec{v}}{c^2} \cdot \vec{r}\right) & t = \gamma\left(t' + \frac{\vec{v}}{c^2} \cdot \vec{r}'\right) \end{array} \Rightarrow \begin{cases} \vec{r}_{\perp} = \vec{r}'_{\perp} \\ \vec{r}_{\parallel} = \gamma(\vec{r}'_{\parallel} + \vec{v}t') \\ t = \gamma\left(t' + \frac{\vec{v}}{c^2} \cdot \vec{r}'\right) \end{cases}$$

Furthermore, we can obtain the Lorentz transformation of ∇

$$\begin{aligned} \frac{\partial}{\partial \vec{r}'_{\perp}} &= \frac{\partial}{\partial \vec{r}_{\perp}} \\ \frac{\partial}{\partial \vec{r}'_{\parallel}} &= \frac{\partial}{\partial \vec{r}_{\parallel}} \frac{\partial \vec{r}_{\parallel}}{\partial \vec{r}'_{\parallel}} + \frac{\partial}{\partial \vec{r}_{\perp}} \frac{\partial \vec{r}_{\perp}}{\partial \vec{r}'_{\parallel}} + \frac{\partial}{\partial t} \frac{\partial t}{\partial \vec{r}'_{\parallel}} \\ &= \frac{\partial}{\partial \vec{r}_{\parallel}} \gamma + \frac{\partial}{\partial \vec{r}_{\perp}} 0 + \frac{\partial}{\partial t} \gamma \frac{\vec{v}}{c^2} \\ &= \gamma \left(\frac{\partial}{\partial \vec{r}_{\parallel}} + \frac{\vec{v}}{c^2} \frac{\partial}{\partial t} \right) \\ \frac{\partial}{\partial t'} &= \frac{\partial}{\partial t} \frac{\partial t}{\partial t'} + \frac{\partial}{\partial \vec{r}} \frac{\partial \vec{r}}{\partial t'} = \gamma \left(\frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{r}} \right) \end{aligned}$$

We express the transformations in a compact form

$$\begin{aligned} \nabla'_{\perp} &= \nabla_{\perp} \\ \nabla'_{\parallel} &= \gamma \left(\nabla_{\parallel} + \frac{\vec{v}}{c^2} \frac{\partial}{\partial t} \right) \\ \frac{\partial}{\partial t'} &= \gamma \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) \end{aligned}$$

- (2) Considering the electromagnetic fields of a moving charge

In S frame:

$$\vec{E} = \underbrace{-\nabla\varphi}_{\text{Helmholtz theorem}} - \underbrace{\frac{\partial}{\partial t}\vec{A}}_{\text{Faraday's law [c.f. 7-2-C]}}, \quad \vec{B} = \underbrace{\nabla \times \vec{A}}_{\text{Helmholtz theorem}}$$

In S' frame:

$$\vec{E}' = -\nabla'\varphi' - \frac{\partial}{\partial t'}\vec{A}', \quad \vec{B}' = \nabla' \times \vec{A}'$$

We decompose $\vec{E}'$ and $\vec{B}'$ into $\vec{E}'_{\parallel}$, $\vec{E}'_{\perp}$, $\vec{B}'_{\parallel}$, and $\vec{B}'_{\perp}$ which lie parallel and perpendicular to $\vec{v}$. Thus, we obtain the parallel and perpendicular components of $\vec{E}'$ are

$$\begin{aligned} \vec{E}'_{\parallel} &= -\nabla'_{\parallel}\varphi' - \frac{\partial}{\partial t'}\vec{A}'_{\parallel} \\ \vec{E}'_{\perp} &= -\nabla'_{\perp}\varphi' - \frac{\partial}{\partial t'}\vec{A}'_{\perp} \end{aligned}$$

and the parallel and perpendicular components of $\vec{B}'$ are

$$\begin{aligned} \vec{B}' &= \nabla' \times \vec{A}' \\ &= (\nabla'_{\parallel} + \nabla'_{\perp}) \times (\vec{A}'_{\parallel} + \vec{A}'_{\perp}) \\ &= \underbrace{\nabla'_{\parallel} \times \vec{A}'_{\parallel} + \nabla'_{\perp} \times \vec{A}'_{\perp}}_{\vec{B}'_{\parallel}} + \underbrace{\nabla'_{\parallel} \times \vec{A}'_{\perp} + \nabla'_{\perp} \times \vec{A}'_{\parallel}}_{\vec{B}'_{\perp}} \\ \vec{B}'_{\parallel} &= (\nabla' \times \vec{A}')_{\parallel} = \underbrace{\nabla'_{\parallel} \times \vec{A}'_{\parallel}}_{=0} + \nabla'_{\perp} \times \vec{A}'_{\perp} = \nabla'_{\perp} \times \vec{A}'_{\perp} \\ \vec{B}'_{\perp} &= (\nabla' \times \vec{A}')_{\perp} = \nabla'_{\parallel} \times \vec{A}'_{\perp} + \nabla'_{\perp} \times \vec{A}'_{\parallel} \end{aligned}$$

- (3) Derive the Lorentz transformation of the electromagnetic fields

We deduce the Lorentz transformation of potentials $\vec{A}$ and φ as follows by considering the Lorentz invariance of electrodynamics:

$$\begin{aligned} \vec{A}'_{\perp} &= \vec{A}_{\perp} \\ \vec{A}'_{\parallel} &= \gamma \left(\vec{A}_{\parallel} - \frac{\vec{v}}{c^2} \varphi \right) \\ \varphi' &= \gamma (\varphi - \vec{v} \cdot \vec{A}) \end{aligned}$$

For each term of the electromagnetic fields

1. $\vec{E}'_{\parallel}$ term:

$$\begin{aligned}
\vec{E}'_{\parallel} &= -\nabla'_{\parallel} \varphi' - \frac{\partial}{\partial t'} \vec{A}'_{\parallel} \\
&= -\gamma \left(\nabla_{\parallel} + \frac{\vec{v}}{c^2} \frac{\partial}{\partial t} \right) \gamma (\varphi - \vec{v} \cdot \vec{A}) - \gamma \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) \gamma \left(\vec{A}_{\parallel} - \frac{\vec{v}}{c^2} \varphi \right) \\
&= -\gamma^2 \left(\nabla_{\parallel} \varphi + \frac{\vec{v}}{c^2} \frac{\partial}{\partial t} \varphi - \nabla_{\parallel} (\vec{v} \cdot \vec{A}) - \frac{\vec{v}}{c^2} \frac{\partial}{\partial t} (\vec{v} \cdot \vec{A}) \right) \\
&\quad - \gamma^2 \left(\frac{\partial}{\partial t} \vec{A}_{\parallel} + (\vec{v} \cdot \nabla) \vec{A}_{\parallel} - \frac{\vec{v}}{c^2} \frac{\partial}{\partial t} \varphi - (\vec{v} \cdot \nabla) \frac{\vec{v}}{c^2} \varphi \right) \\
&= -\gamma^2 \left(\nabla_{\parallel} \varphi - \frac{\vec{v}}{c^2} (\vec{v} \cdot \nabla) \varphi \right) - \gamma^2 \left(\frac{\partial}{\partial t} \vec{A}_{\parallel} - \frac{\vec{v}}{c^2} \frac{\partial}{\partial t} (\vec{v} \cdot \vec{A}) \right) \\
&\quad + \gamma^2 \left(\nabla_{\parallel} (\vec{v} \cdot \vec{A}) - (\vec{v} \cdot \nabla) \vec{A}_{\parallel} \right)
\end{aligned}$$

Since $\vec{v}$ is a constant vector, we obtain

$$\begin{aligned}
\underbrace{\vec{v} \cdot \vec{A}}_{\text{inner product}} &= \vec{v} \cdot \vec{A}_{\parallel} \Rightarrow \vec{v} (\vec{v} \cdot \vec{A}) = v \hat{v} (v A_{\parallel}) = v^2 \vec{A}_{\parallel} \\
\underbrace{\vec{v} \cdot \nabla}_{\text{inner product}} &= \vec{v} \cdot \nabla_{\parallel} \Rightarrow \vec{v} (\vec{v} \cdot \nabla) = v^2 \nabla_{\parallel}
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\vec{E}'_{\parallel} &= -\gamma^2 \left(\nabla_{\parallel} \varphi - \frac{v^2}{c^2} \nabla_{\parallel} \varphi \right) - \gamma^2 \left(\frac{\partial}{\partial t} \vec{A}_{\parallel} - \frac{v^2}{c^2} \frac{\partial}{\partial t} \vec{A}_{\parallel} \right) \\
&\quad + \gamma^2 \left(\nabla_{\parallel} (\vec{v} \cdot \vec{A}_{\parallel}) - (\vec{v} \cdot \nabla_{\parallel}) \vec{A}_{\parallel} \right) \\
&= -\gamma^2 \left(1 - \frac{v^2}{c^2} \right) \nabla_{\parallel} \varphi - \gamma^2 \left(1 - \frac{v^2}{c^2} \right) \frac{\partial}{\partial t} \vec{A}_{\parallel} \\
&\quad + \gamma^2 \left(\nabla_{\parallel} (\vec{v} \cdot \vec{A}_{\parallel}) - (\vec{v} \cdot \nabla_{\parallel}) \vec{A}_{\parallel} \right) \\
&= -\nabla_{\parallel} \varphi - \frac{\partial}{\partial t} \vec{A}_{\parallel} + \gamma^2 \left(\nabla_{\parallel} (\vec{v} \cdot \vec{A}_{\parallel}) - (\vec{v} \cdot \nabla_{\parallel}) \vec{A}_{\parallel} \right)
\end{aligned}$$

Using vector triple product identity:

$$\vec{v} \times \underbrace{(\nabla_{\parallel} \times \vec{A}_{\parallel})}_{=0} = \nabla_{\parallel} (\vec{v} \cdot \vec{A}_{\parallel}) - (\vec{v} \cdot \nabla_{\parallel}) \vec{A}_{\parallel} = 0$$

We obtain

$$\vec{E}'_{\parallel} = -\nabla_{\parallel} \varphi - \frac{\partial}{\partial t} \vec{A}_{\parallel} = \vec{E}_{\parallel}$$

2. $\vec{E}'_{\perp}$ term:

$$\begin{aligned}
\vec{E}'_{\perp} &= -\nabla'_{\perp} \varphi' - \frac{\partial}{\partial t'} \vec{A}'_{\perp} \\
&= -\nabla_{\perp} \gamma \left(\varphi - \vec{v} \cdot \vec{A}_{\parallel} \right) - \gamma \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla_{\parallel} \right) \vec{A}_{\perp} \\
&= \gamma \left(-\nabla_{\perp} \varphi - \frac{\partial}{\partial t} \vec{A}_{\perp} \right) + \gamma \left(\nabla_{\perp} \left(\vec{v} \cdot \vec{A}_{\parallel} \right) - (\vec{v} \cdot \nabla_{\parallel}) \vec{A}_{\perp} \right)
\end{aligned}$$

Using vector triple product identity:

$$\begin{aligned}
\vec{v} \times \left(\nabla_{\perp} \times \vec{A}_{\parallel} \right) &= \nabla_{\perp} \left(\vec{v} \cdot \vec{A}_{\parallel} \right) - \underbrace{(\vec{v} \cdot \nabla_{\perp}) \vec{A}_{\parallel}}_{=0} = \nabla_{\perp} \left(\vec{v} \cdot \vec{A}_{\parallel} \right) \\
\vec{v} \times \left(\nabla_{\parallel} \times \vec{A}_{\perp} \right) &= \nabla_{\parallel} \left(\vec{v} \cdot \vec{A}_{\perp} \right) - \underbrace{(\vec{v} \cdot \nabla_{\parallel}) \vec{A}_{\perp}}_{=0} = -(\vec{v} \cdot \nabla_{\parallel}) \vec{A}_{\perp}
\end{aligned}$$

We obtain

$$\begin{aligned}
\vec{E}'_{\perp} &= \gamma \vec{E}_{\perp} + \gamma \left(\vec{v} \times \left(\nabla_{\perp} \times \vec{A}_{\parallel} \right) + \vec{v} \times \left(\nabla_{\parallel} \times \vec{A}_{\perp} \right) \right) \\
&= \gamma \vec{E}_{\perp} + \gamma \vec{v} \times \left(\nabla_{\perp} \times \vec{A}_{\parallel} + \nabla_{\parallel} \times \vec{A}_{\perp} \right) \\
&= \gamma \vec{E}_{\perp} + \gamma \vec{v} \times \vec{B}_{\perp} \\
&= \gamma \left(\vec{E} + \vec{v} \times \vec{B} \right)_{\perp}
\end{aligned}$$

3. $\vec{B}'_{\parallel}$ term:

$$\vec{B}'_{\parallel} = \nabla'_{\perp} \times \vec{A}'_{\perp}$$

Since

$$\nabla'_{\perp} = \nabla_{\perp} \text{ and } \vec{A}'_{\perp} = \vec{A}_{\perp}$$

we obtain

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel}$$

4. $\vec{B}'_{\perp}$ term:

$$\vec{B}'_{\perp} = \nabla'_{\parallel} \times \vec{A}'_{\perp} + \nabla'_{\perp} \times \vec{A}'_{\parallel}$$

Since

$$\nabla'_{\parallel} = \gamma \left(\nabla_{\parallel} + \frac{\vec{v}}{c^2} \frac{\partial}{\partial t} \right) \text{ and } \vec{A}'_{\parallel} = \gamma \left(\vec{A}_{\parallel} - \frac{\vec{v}}{c^2} \varphi \right)$$

we obtain

$$\begin{aligned}
\vec{B}'_{\perp} &= \gamma \left(\nabla_{\parallel} + \frac{\vec{v}}{c^2} \frac{\partial}{\partial t} \right) \times \vec{A}_{\perp} + \nabla_{\perp} \times \gamma \left(\vec{A}_{\parallel} - \frac{\vec{v}}{c^2} \varphi \right) \\
&= \gamma \left(\nabla_{\parallel} \times \vec{A}_{\perp} + \nabla_{\perp} \times \vec{A}_{\parallel} \right) + \frac{\gamma}{c^2} \left[\vec{v} \times \frac{\partial \vec{A}_{\perp}}{\partial t} - \nabla_{\perp} \times (\vec{v} \varphi) \right] \\
&= \gamma \vec{B}_{\perp} + \frac{\gamma}{c^2} \left[-\vec{v} \times \left(-\frac{\partial \vec{A}_{\perp}}{\partial t} \right) + \vec{v} \times \nabla_{\perp} \varphi \right] \\
&= \gamma \vec{B}_{\perp} - \frac{\gamma}{c^2} \vec{v} \times \left(-\frac{\partial \vec{A}_{\perp}}{\partial t} - \nabla_{\perp} \varphi \right) \\
&= \gamma \vec{B}_{\perp} - \frac{\gamma}{c^2} \vec{v} \times \vec{E}_{\perp} \\
&= \gamma \left(\vec{B} - \frac{\vec{v}}{c^2} \times \vec{E} \right)_{\perp}
\end{aligned}$$

Thus, the Lorentz transformation of electromagnetic fields are

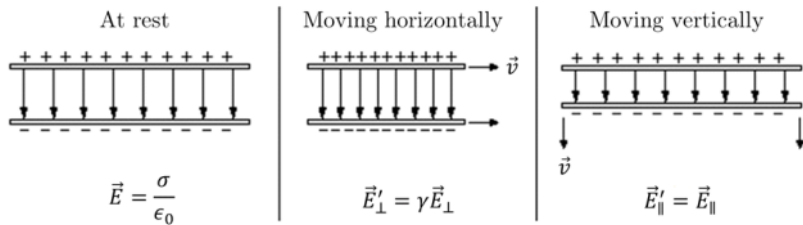
$$\begin{aligned}
\vec{E}'_{\parallel} &= \vec{E}_{\parallel}, & \vec{E}'_{\perp} &= \gamma \left(\vec{E} + \vec{v} \times \vec{B} \right)_{\perp} \\
\vec{B}'_{\parallel} &= \vec{B}_{\parallel}, & \vec{B}'_{\perp} &= \gamma \left(\vec{B} - \frac{\vec{v}}{c^2} \times \vec{E} \right)_{\perp}
\end{aligned}$$

- (4) Suppose S frame exists in which $\vec{B} = 0$ in some region. Then in any other S' frame that moves with velocity $\vec{v}$ relative to S frame, we have

$$\boxed{
\begin{aligned}
\vec{E}'_{\parallel} &= \vec{E}_{\parallel}, & \vec{E}'_{\perp} &= \gamma \vec{E}_{\perp} \\
\vec{B}'_{\parallel} &= 0, & \vec{B}'_{\perp} &= -\gamma \frac{\vec{v}}{c^2} \times \vec{E}_{\perp}
\end{aligned}
} \dots (a)$$

EXAMPLES:

1. Consider a pair of closely spaced charged plates.



ANSWER:

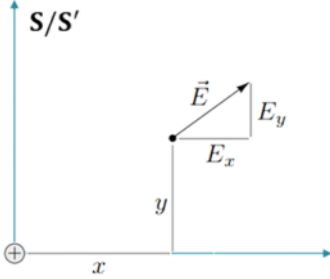
If we put the plates in motion to the right they get length-contracted, but the total charge on each is an invariant so the surface charge density increases by the Lorentz factor, and therefore so does the field strength.

If the plates are moving vertically then the distance between

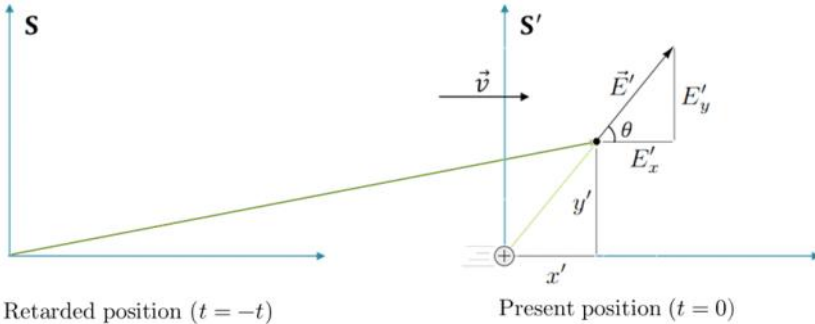
them gets length-contracted, but for closely spaced plates the distance has no effect on the field strength so the field strength is the same as in the original reference frame.

B. GAUSS'S LAW FOR MOVING CHARGES

(1) A moving point charge q



We consider the fields seen by an observer in the S frame when a point charge q moves with a velocity v . The charge is at rest in S' frame, i.e., the field of the point charge in its own frame is entirely electrostatic:



$$E'_x = \frac{q}{4\pi\epsilon_0 r'^2} \cos \theta = \frac{q}{4\pi\epsilon_0} \frac{x'}{(x'^2 + y'^2)^{3/2}}$$

$$E'_y = \frac{q}{4\pi\epsilon_0 r'^2} \sin \theta = \frac{q}{4\pi\epsilon_0} \frac{y'}{(x'^2 + y'^2)^{3/2}}$$

Since $\vec{B}' = 0$ in S' frame, we have, according to equation (a),

$$\vec{E}_{\parallel} = \vec{E}'_{\parallel}, \quad \vec{E}_{\perp} = \gamma \vec{E}'_{\perp}$$

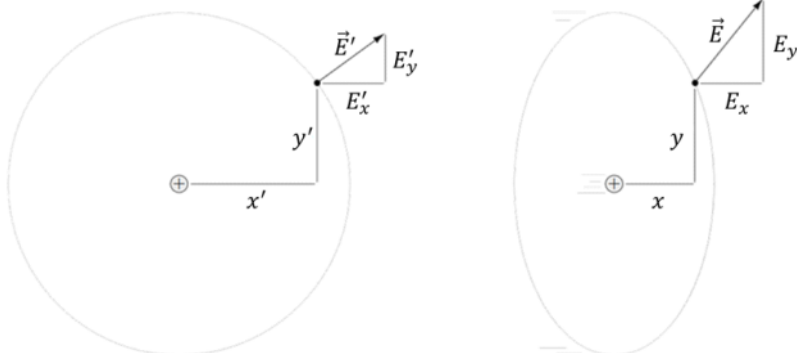
Thus, we obtain

$$E_x = E'_x = \frac{q}{4\pi\epsilon_0} \frac{x'}{(x'^2 + y'^2)^{3/2}}$$

$$E_y = \gamma E'_y = \frac{q}{4\pi\epsilon_0} \frac{\gamma y'}{(x'^2 + y'^2)^{3/2}}$$

Point charge at rest (in S' frame)

Point charge in motion (in S frame)



As the point charge is at rest (in S' frame), every point on the sphere, the field has the same strength and points directly away from the point charge.

As the point charge is in motion to the right (in S frame), the sphere gets length-contracted, flattened in the direction of motion and the components of the field perpendicular to the motion get stretched by the very same Lorentz factor.

Using Lorentz transformation and let $t = 0$:

$$x' = \gamma(x - vt) = \gamma x$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma \left(t - \frac{v}{c^2} x \right) = -\gamma \frac{v}{c^2} x$$

The fields become

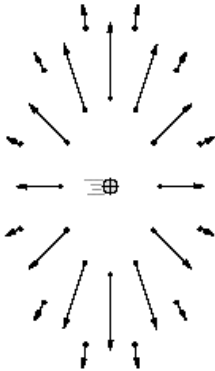
$$E_x = \frac{q}{4\pi\epsilon_0} \frac{x'}{(x'^2 + y'^2)^{3/2}} = \frac{q}{4\pi\epsilon_0} \frac{\gamma x}{(\gamma^2 x^2 + y^2)^{3/2}}$$

$$E_y = \frac{q}{4\pi\epsilon_0} \frac{\gamma y'}{(x'^2 + y'^2)^{3/2}} = \frac{q}{4\pi\epsilon_0} \frac{\gamma y}{(\gamma^2 x^2 + y^2)^{3/2}}$$

We obtain the electric field in the laboratory frame:

$$\begin{aligned}
\vec{E} &= \frac{q}{4\pi\epsilon_0} \frac{\gamma x}{(\gamma^2 x^2 + y^2)^{3/2}} \hat{x} + \frac{q}{4\pi\epsilon_0} \frac{\gamma y}{(\gamma^2 x^2 + y^2)^{3/2}} \hat{y} \\
&= \frac{q}{4\pi\epsilon_0} \frac{1}{\gamma^2 (x^2 + y^2/\gamma^2)^{3/2}} (x\hat{x} + y\hat{y}) \\
&= \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(R^2 \cos^2 \theta + (1 - v^2/c^2) R^2 \sin^2 \theta)^{3/2}} \vec{R} \\
&= \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(1 - v^2/c^2 \sin^2 \theta)^{3/2}} \frac{\hat{R}}{R^2} \dots (b)
\end{aligned}$$

The electric field is radial, but the lines of force are isotropically distributed only for $v = 0$. Along the direction of motion ($\theta = 0$), the field strength is down by a factor $1/\gamma^2$ relative to isotropy, while in the transverse directions ($\theta = \pi/2$) it is larger by a factor of γ .



The field points directly away from the point charge, but it's not the same in all directions: it's weaker in front of and behind the particle, and stronger to the sides

OS:

Further reading: accelerating charge

<https://physics.stackexchange.com/questions/296904/electric-field-associated-with-moving-charge>

(2) Gauss's law for a moving point charge

$$\begin{aligned}
d\Phi &= \vec{E} \cdot d\vec{a} = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(1 - v^2/c^2 \sin^2 \theta)^{3/2}} \frac{\hat{R}}{R^2} \cdot R^2 \sin \theta d\theta d\phi \hat{R} \\
\Phi &= \oint_S d\Phi = \int_0^\pi \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(1 - v^2/c^2 \sin^2 \theta)^{3/2}} \sin \theta d\theta \underbrace{\int_0^{2\pi} d\phi}_{=2\pi}
\end{aligned}$$

Using the integral formula:

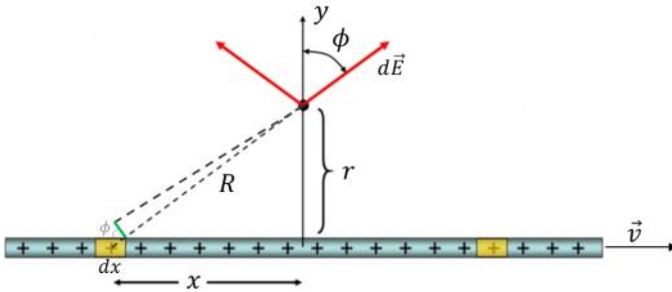
$$\int \frac{\sin \theta d\theta}{(1 - a^2 \sin^2 \theta)^{3/2}} = \frac{-\cos \theta}{(1 - a^2)(1 - a^2 \sin^2 \theta)^{1/2}}$$

We obtain the flux of a closed surface around a moving charge

$$\begin{aligned}\Phi &= \frac{q(1 - v^2/c^2)}{4\pi\epsilon_0} \frac{-\cos \theta}{(1 - v^2/c^2)(1 - v^2/c^2 \sin^2 \theta)^{1/2}} \Bigg|_0^\pi \cdot 2\pi \\ &= \frac{q(1 - v^2/c^2)}{2\epsilon_0} \frac{2}{(1 - v^2/c^2)} \\ &= \frac{q}{\epsilon_0}\end{aligned}$$

(3) A line of moving charges (moving rod)

An infinity long rod carrying uniform charge λ moving with velocity v .



From equation (b), we have

$$\begin{aligned}\phi &= \frac{\pi}{2} - \theta \\ d\vec{E} &= \frac{dq}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(1 - v^2/c^2 \sin^2 \theta)^{3/2}} \frac{\hat{R}}{R^2} \\ &= \frac{dq}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(1 - v^2/c^2 \cos^2 \phi)^{3/2}} \frac{\hat{R}}{R^2} \\ \vec{E} &= \int_{-\pi/2}^{\pi/2} d\vec{E} \cos \phi = \int_{-\pi/2}^{\pi/2} \frac{dq}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(1 - v^2/c^2 \sin^2 \theta)^{3/2}} \frac{\hat{r}}{R^2} \cos \phi\end{aligned}$$

Since

$$\left. \begin{aligned}dq &= \lambda dx \\ R d\phi &= dx \cos \phi \\ r &= R \cos \phi\end{aligned} \right\} \Rightarrow dq = \lambda R \frac{d\phi}{\cos \phi} = \lambda \frac{r}{\cos \phi} \frac{d\phi}{\cos \phi} = \frac{\lambda r d\phi}{\cos^2 \phi}$$

Thus, we obtain

$$\begin{aligned}
\vec{E} &= \int_{-\pi/2}^{\pi/2} \frac{1}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(1 - v^2/c^2 \cos^2 \phi)^{3/2}} \frac{\hat{r}}{r^2/\cos^2 \phi} \cos \phi \frac{\lambda r d\phi}{\cos^2 \phi} \\
&= \int_{-\pi/2}^{\pi/2} \frac{\lambda}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(1 - v^2/c^2 \cos^2 \phi)^{3/2}} \frac{\hat{r}}{r} \cos \phi d\phi \\
&= \int_{-\pi/2}^{\pi/2} \frac{\lambda}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{(1 - v^2/c^2 + v^2/c^2 \sin^2 \phi)^{3/2}} \frac{\hat{r}}{r} \frac{c}{v} d\left(\frac{v}{c} \sin \phi\right)
\end{aligned}$$

Using the integral formula:

$$\int \frac{dt}{(a^2 + t^2)^{3/2}} = \frac{t}{a^2(a^2 + t^2)^{1/2}}$$

We obtain

$$\begin{aligned}
\vec{E} &= \frac{\lambda}{4\pi\epsilon_0} \frac{\hat{r}}{r} \frac{c}{v} \frac{(1 - v^2/c^2) \frac{v}{c} \sin \phi}{(1 - v^2/c^2)(1 - v^2/c^2 + v^2/c^2 \sin^2 \phi)^{1/2}} \bigg|_{-\pi/2}^{\pi/2} \\
&= \frac{\lambda}{4\pi\epsilon_0} \frac{\hat{r}}{r} \frac{(1 - v^2/c^2) 2}{(1 - v^2/c^2)(1 - v^2/c^2 + v^2/c^2)^{1/2}} \\
&= \frac{\lambda}{2\pi\epsilon_0 r} \hat{r}
\end{aligned}$$

(4) Gauss's law for a line of moving charges

The field of the infinity long rod in its own frame is entirely electrostatic, i.e.,

$$\oiint_S \vec{E}' \cdot d\vec{a}' = E' \cdot 2\pi r' l' = \frac{Q}{\epsilon_0} \Rightarrow \vec{E}' = \frac{Q}{2\pi\epsilon_0 r' l'} = \frac{\lambda'}{2\pi\epsilon_0 r'} \hat{r}'$$

Since $\vec{B}' = 0$ in S' frame, we have, according to equation (a),

$$\vec{E}_{\parallel} = \vec{E}'_{\parallel}, \quad \vec{E}_{\perp} = \gamma \vec{E}'_{\perp}$$

Thus, we obtain

$$\vec{E} = \gamma \vec{E}' = \frac{\gamma \lambda'}{2\pi\epsilon_0 r'} \hat{r} \stackrel{?}{=} \frac{\lambda}{2\pi\epsilon_0 r} \hat{r}$$

Since S' frame moves with velocity v with respect to S frame,

$$r = r'$$

the distance between charges in the rod as seen in S frame is contracted by

$$l = \frac{l'}{\gamma}$$

The linear density of positive charge in this frame is correspondingly

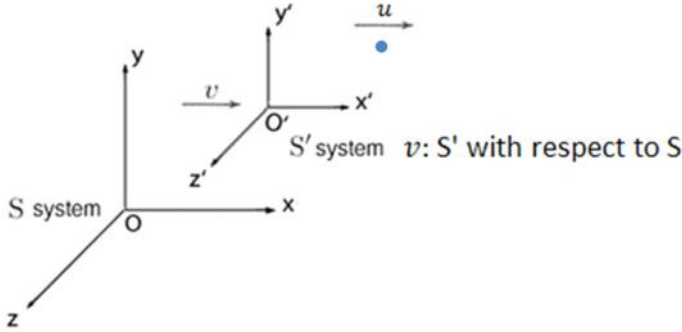
$$\lambda = \frac{Q}{l} = \frac{Q}{l'/\gamma} = \gamma \frac{Q}{l'} = \gamma \lambda'$$

Thus, we obtain

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r} \hat{r}$$

C. LORENTZ FORCE BETWEEN MOVING TEST CHARGE AND OTHER CHARGES

(1) The Lorentz transformation of the force



$$\begin{aligned}\vec{F}'_{\parallel} &= \frac{d\vec{p}'_{\parallel}}{dt'} = \frac{d\vec{p}'_{\parallel}}{dt} \frac{dt}{dt'} = \frac{d\vec{p}'_{\parallel}}{dt} \bigg/ \frac{dt}{dt'} \\ \vec{F}'_{\perp} &= \frac{d\vec{p}'_{\perp}}{dt'} = \frac{d\vec{p}'_{\perp}}{dt} \frac{dt}{dt'} = \frac{d\vec{p}'_{\perp}}{dt} \bigg/ \frac{dt}{dt'}\end{aligned}$$

Since

$$t' = \gamma \left(t - \frac{\vec{v} \cdot \vec{r}}{c^2} \right) \Rightarrow \frac{dt'}{dt} = \gamma \left(1 - \frac{\vec{v} \cdot \vec{u}}{c^2} \right)$$

Thus, we obtain

$$\begin{aligned}\vec{F}'_{\parallel} &= \frac{\frac{d}{dt} \gamma \left(\vec{p}_{\parallel} - \frac{\vec{v}}{c^2} E \right)}{\gamma \left(1 - \frac{\vec{v} \cdot \vec{u}}{c^2} \right)} = \frac{\frac{d\vec{p}_{\parallel}}{dt} - \frac{\vec{v}}{c^2} \frac{dE}{dt}}{\left(1 - \frac{\vec{v} \cdot \vec{u}}{c^2} \right)} = \frac{\vec{F}_{\parallel} - \frac{\vec{v}}{c^2} \frac{dE}{dt}}{\left(1 - \frac{\vec{v} \cdot \vec{u}}{c^2} \right)} \\ \vec{F}'_{\perp} &= \frac{\frac{d\vec{p}_{\perp}}{dt}}{\gamma \left(1 - \frac{\vec{v} \cdot \vec{u}}{c^2} \right)} = \frac{\vec{F}_{\perp}}{\gamma \left(1 - \frac{\vec{v} \cdot \vec{u}}{c^2} \right)}\end{aligned}$$

From the work-energy theorem,

$$E = \int \vec{F} \cdot d\vec{r} = \int \frac{d\vec{p}}{dt} \cdot d\vec{r} = \int \frac{d\vec{p}}{dt} \cdot \frac{d\vec{r}}{dt} dt = \int \frac{d\vec{p}}{dt} \cdot \vec{u} dt$$

we have

$$\frac{dE}{dt} = \frac{d\vec{p}}{dt} \cdot \vec{u} = \vec{F} \cdot \vec{u} = (\vec{F}_{\parallel} + \vec{F}_{\perp}) \cdot \vec{u}$$

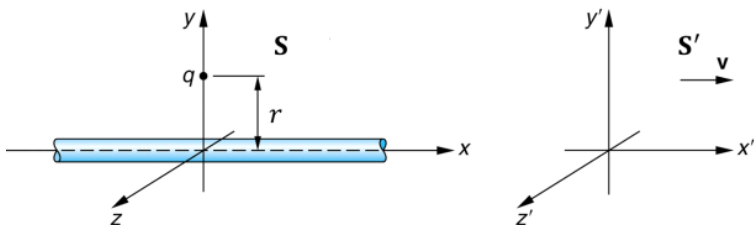
Thus, we obtain

$$\begin{aligned}
 \vec{F}'_{\parallel} &= \frac{\vec{F}_{\parallel} - \frac{\vec{v}}{c^2} \left((\vec{F}_{\parallel} + \vec{F}_{\perp}) \cdot \vec{u} \right)}{\left(1 - \frac{\vec{v} \cdot \vec{u}}{c^2} \right)} \\
 &= \vec{F}_{\parallel} \frac{\left(1 - \frac{\vec{v} \cdot \vec{u}}{c^2} \right)}{\left(1 - \frac{\vec{v} \cdot \vec{u}}{c^2} \right)} - \frac{\frac{\vec{v}}{c^2}}{\left(1 - \frac{\vec{v} \cdot \vec{u}}{c^2} \right)} \vec{F}_{\perp} \cdot \vec{u} \\
 &= \vec{F}_{\parallel} - \frac{\frac{\vec{v}}{c^2}}{\left(1 - \frac{\vec{v} \cdot \vec{u}}{c^2} \right)} \vec{F}_{\perp} \cdot \vec{u}
 \end{aligned}$$

If the S frame is the rest frame of the particle, then $\vec{u} = 0$ and we have

$$\begin{aligned}
 \vec{F}'_{\parallel} &= \vec{F}_{\parallel} \\
 \vec{F}'_{\perp} &= \frac{\vec{F}_{\perp}}{\gamma}
 \end{aligned}$$

(2) Stationary rod and stationary test charge q



A test charge q at rest a distance r from an infinity long rod carrying uniform charge λ . The field of the infinity long rod in S frame is entirely electrostatic, i.e.,

$$\oint_S \vec{E} \cdot d\vec{a} = E \cdot 2\pi r l = \frac{Q}{\epsilon_0} \Rightarrow \vec{E} = \frac{Q}{2\pi\epsilon_0 r l} = \frac{\lambda}{2\pi\epsilon_0 r} \hat{r}$$

The Lorentz force on the test charge is

$$\vec{F} = q\vec{E} = \frac{q\lambda}{2\pi\epsilon_0 r} \hat{r}$$

Now consider an observer moving to the right with velocity v . In S' frame, both the test charge and the rod move to the left, and

$$r' = r$$

Since $\vec{B} = 0$ in S frame, we have, according to equation (a),

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}, \quad \vec{E}'_{\perp} = \gamma \vec{E}_{\perp}$$

$$\vec{B}'_{\parallel} = 0, \quad \vec{B}'_{\perp} = -\gamma \frac{\vec{v}}{c^2} \times \vec{E}_{\perp}$$

Thus, in S' frame, we obtain

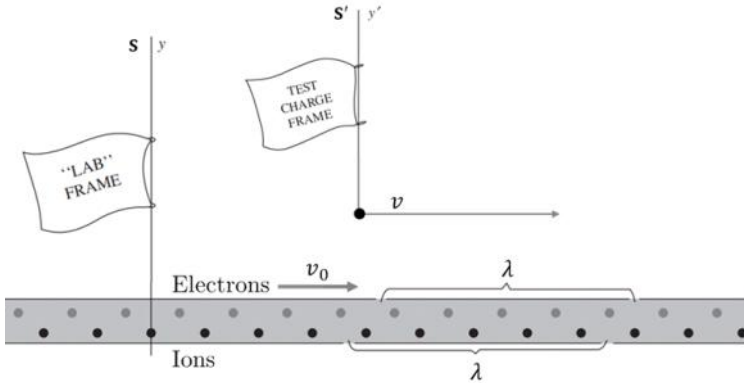
$$\vec{E}' = \gamma \vec{E} = \frac{\gamma \lambda}{2\pi\epsilon_0 r} \hat{r}$$

$$\vec{B}' = -\gamma \frac{\vec{v}}{c^2} \times \vec{E} = -\frac{\gamma \lambda}{2\pi\epsilon_0 c^2 r} \vec{v} \times \hat{r} = -\frac{\gamma \lambda v}{2\pi\epsilon_0 c^2 r} \hat{z}$$

and the Lorentz force is

$$\begin{aligned} \vec{F}' &= q\vec{E}' + q\vec{v} \times \vec{B}' \\ &= \frac{\gamma q \lambda}{2\pi\epsilon_0 r} \hat{r} + q\vec{v} \times \left(-\frac{\gamma \lambda v}{2\pi\epsilon_0 c^2 r} \hat{z} \right) \\ &= \frac{\gamma q \lambda}{2\pi\epsilon_0 r} \hat{r} - \frac{\gamma q \lambda v^2}{2\pi\epsilon_0 r c^2} \vec{r} \\ &= \frac{\gamma q \lambda}{2\pi\epsilon_0 r} \left(1 - \frac{v^2}{c^2} \right) \hat{r} \\ &= \frac{q \lambda}{2\pi\epsilon_0 r} \frac{1}{\gamma} \hat{r} \\ &= \frac{\vec{F}}{\gamma} \end{aligned}$$

- (3) The Lorentz force between a moving charge and other moving charges.



In the test charge frame (S'), the positive ions are moving with velocity $-v$. The distance between ions as seen in S' frame is contracted by

$$l' = \frac{l}{\gamma}$$

The linear density of **positive ions** is correspondingly

$$\lambda'_{\text{ion}} = \frac{Q}{l'} = \frac{Q}{l/\gamma} = \gamma \frac{Q}{l} = \gamma \lambda$$

Electrons are already moving with velocity v_0 in the lab frame (S). Using addition of velocities, the velocity of electrons in S' frame is

$$v'_0 = \frac{v_0 - v}{1 - v_0 v / c^2}$$

The Lorentz factor of electrons in S' frame is

$$\gamma'_0 = \frac{1}{\sqrt{1 - v'^2_0 / c^2}}, \quad \gamma_0 = \frac{1}{\sqrt{1 - v^2_0 / c^2}}$$

Thus, the linear density of **electrons** is correspondingly

$$\lambda'_{\text{electron}} = \gamma'_0 \frac{\lambda}{\gamma_0}$$

The total linear density of charge in S' frame is

$$\lambda' = \lambda'_{\text{ion}} - \lambda'_{\text{electron}} = \gamma \lambda - \gamma'_0 \frac{\lambda}{\gamma_0}$$

Since

$$\begin{aligned} \frac{1}{\sqrt{1 - \frac{v'^2_0}{c^2}}} &= \frac{1}{\sqrt{1 - \frac{1}{c^2} \left(\frac{v_0 - v}{1 - v_0 v / c^2} \right)^2}} \\ &= \frac{1}{\sqrt{\frac{(1 - v_0 v / c^2)^2}{(1 - v_0 v / c^2)^2} - \frac{(v_0 - v)^2 / c^2}{(1 - v_0 v / c^2)^2}}} \\ &= \frac{1}{\sqrt{\frac{1 - \frac{v_0^2}{c^2} - \frac{v^2}{c^2} + \frac{v_0^2 v^2}{c^4}}{(1 - v_0 v / c^2)^2}}} \\ &= \frac{1}{\sqrt{\frac{(1 - v_0^2 / c^2)(1 - v^2 / c^2)}{(1 - v_0 v / c^2)^2}}} \\ &= \frac{1}{\underbrace{\sqrt{1 - v_0^2 / c^2}}_{=\gamma_0}} \underbrace{\frac{1}{\sqrt{1 - v^2 / c^2}}}_{=\gamma} \left(1 - \frac{v_0 v}{c^2} \right) \end{aligned}$$

we obtain

$$\lambda' = \gamma\lambda - \gamma'_0 \frac{\lambda}{\gamma_0} = \gamma\lambda - \gamma_0\gamma \left(1 - \frac{v_0 v}{c^2}\right) \frac{\lambda}{\gamma_0} = \gamma\lambda \frac{v_0 v}{c^2}$$

Thus, in S' frame, $\lambda' \neq 0$ implies that the total charge is not neutral around the test charge. The field of the wire in S' frame is

$$\vec{E}' = \frac{\lambda'}{2\pi\epsilon_0 r'} \hat{r}' = \frac{\gamma\lambda \frac{v_0 v}{c^2}}{2\pi\epsilon_0 r'} \hat{r}' = \frac{\gamma\lambda v_0 v}{2\pi\epsilon_0 c^2 r'} \hat{r}'$$

The test charge will therefore experience a Lorentz force

$$\vec{F}' = q\vec{E}' = \frac{q\gamma\lambda v_0 v}{2\pi\epsilon_0 c^2 r'} \hat{r}'$$

The force on the moving test charge, measured in S frame, is

$$\vec{F} = \frac{\vec{F}'}{\gamma} = -\frac{\overbrace{q\lambda v_0 v}^{=I}}{2\pi\epsilon_0 c^2 r'} \hat{r}' = -\frac{qvI}{2\pi\epsilon_0 c^2 r} \hat{r}$$

Since

$$c^2 = \frac{1}{\mu_0 \epsilon_0}$$

thus, we obtain the force

$$\vec{F} = -\frac{\mu_0 qvI}{2\pi r} \hat{r} = qv \left(\frac{\mu_0 I}{2\pi r} \right) (-\hat{y})$$

Let $\hat{x} \times \hat{z} = -\hat{y}$ and $\vec{v} = v\hat{x}$. The force, in S frame, becomes

$$\vec{F} = qv\hat{x} \times \left(\frac{\mu_0 I}{2\pi r} \right) \hat{z} = q\vec{v} \times \frac{\mu_0 I}{2\pi r} \hat{z} = q\vec{v} \times \vec{B}$$

where

$$\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{z}$$

Conclusion: if a charge is stationary, it only produces an electric field, but when viewing the charge from a frame that is moving relative to the charge, a magnetic field is also produced.

NOTE:

An electric field for someone in the stationary frame, might appear as a magnetic field for someone else in the moving frame, because electric and magnetic fields are not invariant under the Lorentz transformation. That is, two observers may disagree on whether a field "looks" electric or magnetic.